

Turbulent compressible fluid: renormalization group analysis, scaling regimes, and anomalous scaling of advected scalar fields

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A model of fully developed turbulence of a *compressible* fluid, based on the stochastic Navier-Stokes equation, is studied by means of the field theoretic renormalization group. This model has been already analyzed earlier in [Theor. Math. Phys., **110**, 3 (1997)]. Its scaling properties are related to fixed points of the renormalization group equations. Our aim here is to study a possibility of existence of other scaling regimes and to analyze an opportunity of crossover between them. This may occur in some other space dimensions than previously considered, namely in $d = 4$. A new regime arises there and then continuously moves into $d = 3$. Our calculations show that there really exists an additional fixed point that governs scaling behavior. Turbulent advection of a passive scalar (density) field by this velocity ensemble is considered as well. It is shown that various correlation functions of the scalar field demonstrate anomalous scaling behavior in the inertial-convective range. The corresponding anomalous exponents, identified as scaling dimensions of certain composite fields, can be systematically calculated as a series in y (the exponent, connected with random force) and $\varepsilon = 4 - d$. All calculations are performed in the leading one-loop approximation.

Keywords: anomalous scaling, passive vector advection, turbulence, renormalization group

I. INTRODUCTION

Understanding of fully developed turbulence is a complex, rich, and challenging problem. Among the most important features of such behavior are cascades of energy and intermittency. The former one brings energy from large scales, responsible for a creation of turbulence, toward smaller scales, in which viscosity begins to play a major role and dissipation effects dominate, whereas the latter one, being itself an irregular alternation of phases of some dynamics, in fact means that very rare configurations of a system give the most significant contributions to statistical distributions. For turbulence this fact is revealed in anomalous scaling, which implies singular behavior of various statistical quantities as functions of the integral turbulence scales [1].

Another very interesting problem is a turbulent advection of an impurity field. Both natural experiments and numerical simulations suggest that deviations from the classical Kolmogorov theory are even more strongly pronounced for passively advected fields than for the velocity field itself [2–5]. A turbulent environment in such models may be introduced by some “synthetic” velocity field with prescribed statistics or by stochastic Navier-Stokes equation [6]. Models of the former type are more tractable from a mathematical point of view, whereas the latter models bear a closer resemblance to the real world.

A very powerful theoretical framework, which has

worked very well in statistical physics in order to investigate different models of critical phenomena, characterized by scaling behavior, is the field theoretic renormalization group (RG) and the operator product expansion (OPE); it has become an indispensable tool when dealing with systems where no clear separation of scales occurs, see the monographs [7–11]. In the RG+OPE scenario, an anomalous scaling emerges as a consequence of the existence of some composite fields (“composite operators” in the quantum-field terminology) in the model with negative scaling dimensions [12]. The main advantage of this approach with respect to the turbulence is that it is based on a microscopic model and allows one to construct a systematic perturbation expansion for the anomalous exponents, similar to the famous ε -expansion in theory of critical phenomena. The procedure has been successful in verifying Kolmogorov scaling and provides an efficient theoretical tool for calculation of universal quantities [8]. In a number of papers the RG+OPE approach has been applied to the case of passive advection by Kraichnan’s ensemble (velocity field is taken isotropic, Gaussian, not correlated in time, having a power-like correlation function, and fluid assumed to be incompressible), see [13–15], and by its numerous generalizations: large-scale anisotropy, helicity, compressibility, finite correlation time, non-Gaussianity, and a more general form of the nonlinearity [16–26]. A similar approach can be generalized as well to the case of non-Gaussian velocity field, governed by the stochastic Navier-Stokes equation – both for the scaling behavior of velocity field itself, and for an advection of passive fields by this ensemble [27–30].

Up to now a majority of works on fully developed turbulence have been concerned with an incompressible fluid. Nevertheless, several results for compressible flu-

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ids have also been obtained [31–41]. All of them hints at large influence of compressibility both to velocity field itself and passively advected quantities. In particular, the transition from a turbulent to a certain purely chaotic state may occur at large degrees of compressibility [36]. In other papers corrections in the Mach number to the incompressible scaling regime were studied [42–44]. Thus, obtained corrections become arbitrarily large and destroy the incompressible scaling regime for fixed Mach numbers and large distances, what can be explained by an existence of crossover to another, yet unknown regime. The works [45–47] were devoted to a compressible fluid itself. The results are rather controversial, particularly the model, considered in [46], appears to be in fact unrenormalizable. From a general point of view further investigations of compressibility are therefore called for.

In this paper we present an application of the field theoretic renormalization group onto the scaling regimes of a compressible fluid, whose behavior is governed by a proper generalization of the stochastic Navier Stokes equation [45]. A resulting stationary scaling regime in this approach is associated with an infrared (IR) attractive fixed point of the corresponding multiplicatively renormalizable field theoretic model. In a similar fashion as has been done for an incompressible case [48–50], the double expansion scheme is employed. Here the formal expansion parameters are y , which describes the scaling behavior of a random force, and $\varepsilon = 4 - d$, i.e., a deviation from the space dimension $d = 4$.

Following a procedure of previous works [29, 45, 51], first the stochastic Navier-Stokes equation is discussed. After establishing an existence of necessary fixed points of Navier-Stokes equation, the advection of scalar fields is explored. Since RG functions of the parameters, entering into Navier-Stokes equation, do not depend on the parameters, connected with advection-diffusion equation, this is a possible and probably the easiest way.

The paper is organized as follows.

In Sec. II a detailed description of the stochastic Navier-Stokes equation for a compressible fluid is given. Sec. III is devoted to field theoretic formulation of the model and the corresponding diagrammatic technique. In particular, possible types of divergent Green functions at $d = 3$ and $d = 4$ and the necessity of introducing a new coupling constant at $d = 4$ are discussed. In Sec. IV the renormalizability of the model is established and one-loop explicit expressions for the renormalization constants are derived. Much attention is paid here to technical aspects of performed calculations. In Sec. V the RG functions (anomalous dimensions and β functions) are derived and examined. IR asymptotic behavior, obtained as a result of solution of RG equations, is discussed. It is shown that, depending on two exponents y and ε , the RG equations possess an IR attractive fixed point, which implies existence of a scaling regime in the inertial range. The corresponding scaling dimensions of all fields and parameters of the model are presented.

In Sec. VI advection of a passive scalar (density) field

by compressible velocity field, which obeys Navier-Stokes equation, is analyzed. Field theoretic formulation of the full model is presented. It is shown that the full model is multiplicatively renormalizable; the existence of a scaling regime in the IR range is established. The renormalization of composite operators is carried out. An inertial-range behavior of various correlation functions is studied by means of the OPE. It is shown that leading terms of the inertial-range behavior are determined by the contributions of the operators built solely of the scalar fields itself. As a result, the IR behavior of the pair correlation functions of the composite operators is power-like with negative critical dimensions – the situation, called anomalous scaling, occurs.

Sec. VII is reserved for conclusion.

II. DESCRIPTION OF THE MODEL

The Navier-Stokes equation for a viscid compressible fluid can be written in the following form [52]:

$$\rho \nabla_t v_i = \nu_0 [\delta_{ik} \partial^2 - \partial_i \partial_k] v_k + \mu_0 \partial_i \partial_k v_k - \partial_i p + \eta_i, \quad (2.1)$$

where the differential operator on the right hand side

$$\nabla_t = \partial_t + v_k \partial_k \quad (2.2)$$

is the Lagrangian (convective) derivative, ρ is a fluid density field, v_i is a velocity field, $\partial_t = \partial/\partial t$, $\partial_i = \partial/\partial x_i$, $\partial^2 = \partial_i \partial_i$ is the Laplace operator, p is a pressure field, and η_i is a density of an external force per unit volume. The fields v_i , η_i , ρ and p depend on $x = (t, \mathbf{x})$ with $\mathbf{x} = (x_1, x_2, \dots, x_d)$, where d is a dimensionality of space. The constants ν_0 and μ_0 are two independent molecular viscosity coefficients [52]; in (2.1) we have explicitly separated the transverse and longitudinal components of the viscous term. Summations over repeated vector indices are always implied in this work.

To get a closed system of equations, the model (2.1) must be augmented by two additional equations, namely a continuity equation and an equation of state between deviations δp and $\delta \rho$ from the equilibrium values. They explicitly read

$$\partial_t \rho + \partial_i (\rho v_i) = 0 \quad (2.3)$$

and

$$\delta p = c_0^2 \delta \rho. \quad (2.4)$$

In order to obtain the renormalizable field, a theoretic model expression (2.1) has to be divided by ρ , and fluctuations in viscous terms have to be neglected [44]. Further, by using expressions (2.3) and (2.4) the problem can be recast in the form of two coupled equations:

$$\nabla_t v_i = \nu_0 [\delta_{ik} \partial^2 - \partial_i \partial_k] v_k + \mu_0 \partial_i \partial_k v_k - \partial_i \phi + f_i, \quad (2.5)$$

$$\nabla_t \phi = -c_0^2 \partial_i v_i. \quad (2.6)$$

Here a new scalar field $\phi = \phi(x)$ is related to the density fluctuations via the relation $\phi = c_0^2 \ln(\rho/\bar{\rho})$. A parameter c_0 is an adiabatic speed of sound, $\bar{\rho}$ denotes the mean value of ρ , and $f_i = f_i(x)$ is a density of the external force per unit mass.

In the stochastic formulation of the problem the turbulence is modeled by an external force – it is assumed to be a random variable, which mimics the input of energy into the system from the outer large scale L . Its precise form is believed to be unimportant and is usually considered to be a random Gaussian variable with zero mean and prescribed correlation function [7]. For the use of the standard RG technique this correlator must exhibit a power law asymptotic behavior at large wave numbers [8, 53]. In the case of compressible fluid it should be naturally augmented with a non-transverse component, hence the simplest way is to choose it in the form [45]

$$\langle f_i(t, \mathbf{x}) f_j(t', \mathbf{x}') \rangle = \frac{\delta(t - t')}{(2\pi)^d} \int_{k>m} d^d \mathbf{k} D_{ij}(\mathbf{k}) e^{i\mathbf{k} \cdot (\mathbf{x} - \mathbf{x}')}, \quad (2.7)$$

where the argument is given by

$$\tilde{D}_{ij}(\mathbf{k}) = g_{10} \nu_0^3 k^{4-d-y} \left\{ P_{ij}(\mathbf{k}) + \alpha Q_{ij}(\mathbf{k}) \right\}. \quad (2.8)$$

Here, $P_{ij}(\mathbf{k}) = \delta_{ij} - k_i k_j / k^2$ and $Q_{ij}(\mathbf{k}) = k_i k_j / k^2$ are the transverse and longitudinal projectors, $k = |\mathbf{k}|$, amplitude α is a free parameter, an amplitude g_{10} is a cou-

pling constant (formal expansion parameter in the ordinary perturbation theory); the relation $g_0 \sim \Lambda^y$ sets in the typical ultraviolet (UV) momentum scale Λ , which is a reciprocal of the dissipation length scale. A parameter $m = L^{-1}$ provides an infrared regularization; its precise form is unessential and the sharp cut-off is the simplest choice for calculation purposes. An exponent y provides the UV regularization and therefore plays a role of a formally small expansion parameter [7]. The most realistic (physical) value is obtained in the limit $y \rightarrow 4$, when the function in (2.8) can be interpreted as power-like representation of the Dirac function $\delta(\mathbf{k})$: physically it corresponds to the idealized picture of the energy input from infinitely large scales. The Galilean invariance for the model (2.1) is ensured when the function (2.8) is delta-correlated in time [8].

III. FIELD THEORETIC FORMULATION OF THE MODEL

A. Action functional and Feynman rules

According to the general theorem [7, 9], the stochastic problem (2.5) – (2.8) is equivalent to the field theoretic model with a doubled set of fields $\Phi = \{v_i, v'_i, \phi, \phi'\}$ and De Dominicis-Janssen action functional, written in a compact form as

$$\begin{aligned} \mathcal{S}(\Phi) = & \frac{v'_i D_{ik} v'_k}{2} + v'_i \left\{ -\partial_t v_i - v_j \partial_j v_i + \nu_0 [\delta_{ik} \partial^2 - \partial_i \partial_k] v_k + u_0 \nu_0 \partial_i \partial_k v_k - \partial_i \phi \right\} \\ & + \phi' [-\partial_t \phi + v_j \partial_j \phi + v_0 \nu_0 \partial^2 \phi - c_0^2 (\partial_i v_i)], \end{aligned} \quad (3.1)$$

where D_{ik} is the correlation function (2.8). Here a condensed notation has been employed, in which integrals over the spatial variable \mathbf{x} and the time variable t , as well as summation over repeated indices, are implicitly assumed. Moreover, we have introduced a new dimensionless parameter $u_0 = \mu_0 / \nu_0 > 0$ and introduced a new term $v_0 \nu_0 \phi' \partial^2 \phi$ with another positive dimensionless parameter v_0 , which is needed to ensure multiplicative renormalizability. The action (3.1) is amenable to the standard methods of the quantum field theory, such as Feynman diagrammatic technique and renormalization group procedure.

In a standard approach, if quantum field methods are applied to the stochastic differential equations, the space dimension d plays a passive role and an actual perturbative parameter is y ; for more details see the monographs [7, 8]. Our approach closely follows the analysis of the incompressible Navier-Stokes equation near space dimension $d = 2$ [48–50]. In this case an additional di-

vergence appear in the Green function $v'v'$. It can be absorbed by a suitable local counterterm $v'_i \partial^2 v'_i$, and a regular expansion in both y and $\varepsilon' = d - 2$ must be constructed. Up to now the present model (3.1) has been investigated in fixed space dimension $d = 3$, for which the action (3.1) contains all terms that can be generated during the renormalization procedure [29, 30, 45]. However, from the dimensional analysis (see below) it follows that in $d = 4$ an additional divergence appears in a similar fashion in the Green function $v'v'$. Therefore, to keep the model renormalizable in $d = 4$, the kernel function $\tilde{D}_{ij}(\mathbf{k})$ in (2.7) has to be generalized into the function $D_{ij}(\mathbf{k})$, where

$$D_{ij}(\mathbf{k}) = g_{10} \nu_0^3 k^{4-d-y} \left\{ P_{ij}(\mathbf{k}) + \alpha Q_{ij}(\mathbf{k}) \right\} + g_{20} \nu_0^3 \delta_{ij}. \quad (3.2)$$

A new term on the right hand side with an additional coupling constant g_{20} absorbs divergent contributions from $v'v'$. In contrast to the two-dimensional incompressible

case [48] no momentum dependence is needed here.

The field theoretic formulation (3.1) means that various correlation and response functions of the original stochastic problem are represented by functional averages over the full set of fields with functional weight $\exp \mathcal{S}(\Phi)$, and in this sense they can be viewed as Green functions of the field theoretic model [7, 9].

The perturbation theory of the model can be expressed according to the usual Feynman diagrammatic expansion [7, 9]. Bare propagators are read off from the inverse matrix of the Gaussian (free) part of the action functional, while a nonlinear part of the differential equation defines the interaction vertices. A graphical representation of the propagator functions is depicted in Fig. 1, and of the vertices – in Fig. 2. From (3.1) it follows that the Feynman diagrammatic technique for this model contains two vertices, $-v'(v\partial)v$ and $-\phi'(v\partial)\phi$. The propagator functions in the frequency-momentum representation read

$$\begin{aligned}
 \langle vv' \rangle_0 &= \langle v'v \rangle_0^* = P(\mathbf{k})\epsilon_1^{-1} + Q(\mathbf{k})\epsilon_3 R^{-1}, \\
 \langle vv \rangle_0 &= P(\mathbf{k}) \frac{d_1^f}{|\epsilon_1|^2} + Q(\mathbf{k}) d_2^f \left| \frac{\epsilon_3}{R} \right|^2, \\
 \langle \phi v' \rangle_0 &= \langle v' \phi \rangle_0^* = -\frac{ic_0^2 \mathbf{k}}{R}, \quad \langle v \phi' \rangle_0 = \langle \phi' v \rangle_0^* = -\frac{i\mathbf{k}}{R}, \\
 \langle \phi \phi' \rangle_0 &= \langle \phi' \phi \rangle_0^* = \frac{\epsilon_2}{R}, \quad \langle \phi \phi \rangle_0 = \frac{c_0^4 k^2 d_2^f}{|R|^2}, \\
 \langle v \phi \rangle_0 &= \langle \phi v \rangle_0^* = \frac{ic_0^2 d_2^f \epsilon_3 \mathbf{k}}{|R|^2}, \\
 \langle \phi' \phi' \rangle_0 &= \langle v' \phi' \rangle_0 = \langle v' v' \rangle_0 = 0,
 \end{aligned} \tag{3.3}$$

where the vector indices of the fields and the projectors are omitted. For convenience following abbreviations have been used

$$\begin{aligned}
 \epsilon_1 &= -i\omega + \nu_0 k^2, & \epsilon_2 &= -i\omega + u_0 \nu_0 k^2, \\
 \epsilon_3 &= -i\omega + v_0 \nu_0 k^2, & R &= \epsilon_2 \epsilon_3 + c_0^2 k^2
 \end{aligned} \tag{3.4}$$

and

$$\begin{aligned}
 d_1^f &= g_{10} \nu_0^3 k^{4-d-y} + g_{20} \nu_0^3, \\
 d_2^f &= \alpha [g_{10} \nu_0^3 k^{4-d-y}] + g_{20} \nu_0^3.
 \end{aligned} \tag{3.5}$$

In the limit $c_0 \rightarrow \infty$ propagator functions $\langle vv \rangle_0$ and $\langle vv' \rangle_0$ become purely transverse, and all mixed propagators except $\langle \phi v' \rangle_0$ vanish. Moreover, the scalar fields ϕ and ϕ' decouple from the velocity fields \mathbf{v} and \mathbf{v}' – it is impossible to construct a diagram with only velocity fields \mathbf{v} and \mathbf{v}' as external lines, containing internal lines with fields ϕ or ϕ' . Thus, as it should be from physical point of view, the well-known Feynman rules for the incompressible fluid [7, 8] are obtained.

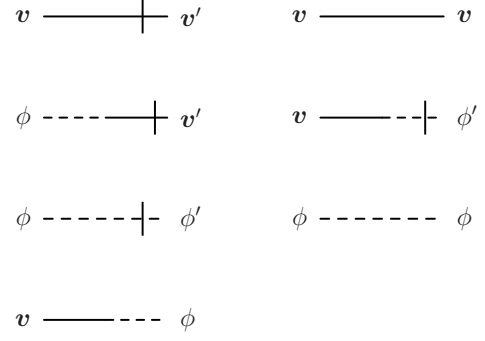


FIG. 1. Graphical representation of the bare propagators in the model (3.1)

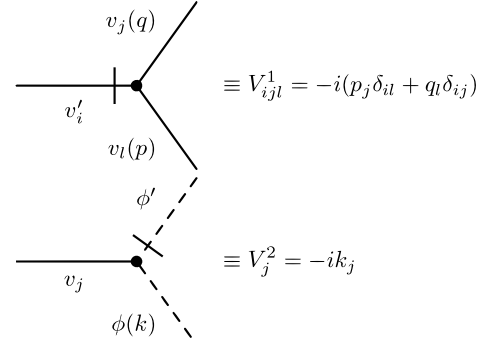


FIG. 2. Graphical representation of the interaction vertices in the model (3.1)

B. Canonical dimensions, UV divergences, and renormalization constants

The ultraviolet renormalizability is very efficiently exhibited in an analysis of the 1-particle irreducible Green functions, called later in notation of [7] as 1-irreducible Green functions. In the case of dynamical models [7, 11] two independent scales have to be introduced: the time scale T and the length scale L . Thus the canonical dimension of any quantity F (a field or a parameter) is described by two numbers, the frequency dimension d_F^ω and the momentum dimension d_F^k , defined such that

$$\begin{aligned}
 d_k^k &= -d_{\mathbf{x}}^k = 1, & d_k^\omega &= d_{\mathbf{x}}^\omega = 0, \\
 d_\omega^\omega &= -d_t^\omega = 1, & d_\omega^k &= d_t^k = 0,
 \end{aligned} \tag{3.6}$$

i.e.,

$$[F] \sim [T]^{-d_F^\omega} [L]^{-d_F^k}. \tag{3.7}$$

The remaining dimensions can be found from the requirement that each term of the action functional be dimensionless, with respect to the momentum and the frequency dimensions separately.

Based on d_F^k and d_F^ω the total canonical dimension $d_F = d_F^k + 2d_F^\omega$ can be introduced, which in the renormalization theory of dynamic models plays the same role as the conventional (momentum) dimension does in static

problems. The choice $\omega \sim k^2$, which provides that all the viscosity and diffusion coefficients in the model are pronounced dimensionless, is not the only possible way for this model. Another choice is to set the speed of sound c_0 dimensionless and consequently obtain that $\omega \sim k$, i.e., $d_F = d_F^k + d_F^\omega$. This variant would mean that we are interested in the asymptotic behavior of the Green functions as $\omega \sim k \rightarrow 0$, in other words, in sound modes in turbulent medium. Even though this problem is very interesting itself, it is not yet accessible to the RG treatment, so we will not discuss it here. The choice $\omega \sim k^2 \rightarrow 0$ is same as in the models of incompressible fluid, where it is the only possibility because the speed of sound is infinite. A similar alternative in dispersion laws exists, for example, in so-called model H of equilibrium dynamical critical behavior, see [7, 11].

The canonical dimensions for the model (3.1) are listed in Table I, including renormalized parameters (without the subscript “0”) and scalar impurity fields θ and θ' , which will appear a bit later. From Table I it follows that this model is logarithmic (the coupling constants $g_{10} \sim [L]^{-y}$ and $g_{20} \sim [L]^{-\varepsilon}$ become dimensionless) at $y = \varepsilon = 0$, so that the UV divergences in the Green functions manifest themselves as poles in y , ε and their linear combinations. Here in accordance with critical phenomena we retain the notation $\varepsilon = 4 - d$.

The total canonical dimension of any 1-irreducible Green function Γ is expressed by the relation

$$\delta_\Gamma = d + 2 - \sum_{\Phi} N_\Phi d_\Phi, \quad (3.8)$$

where N_Φ is the number of the given type of field entering into the function Γ , d_Φ are their total canonical dimensions, and the summation runs over all types of the entering fields Φ [7, 9].

Superficial UV divergences, whose removal requires counterterms, can be present only in the functions Γ for which the formal index of divergence δ_Γ is a non-negative integer. Dimensional analysis should be augmented by the following additional considerations:

- (1) In any dynamical model of the type (3.1) all the 1-irreducible functions without the response fields \mathbf{v}' or ϕ' necessarily contain closed circuits of retarded propagators and therefore vanish identically or at least require no counterterms.
- (2) The field ϕ enters the vertex $\phi'(v\partial)\phi$ only in the form of a spatial derivative, which reduces the real index of divergence:

$$\delta'_\Gamma = \delta_\Gamma - N_\phi. \quad (3.9)$$

In particular this means that the field ϕ enters the counterterms only in the form of the derivative $\partial\phi$ and not all counterterms, allowed by dimensional analysis, are in fact present. For example, for the 1-irreducible function $\langle\phi'\phi\rangle$ one obtains $\delta_\Gamma = 2$, $\delta'_\Gamma = 1$, thus the only possible counterterm is $\phi'\partial^2\phi$, while the structure $\phi'\partial_t\phi$ is in fact forbidden.

- (3) Since the random noise (2.7) is white-in-time, the model (3.1) is Galilean invariant. Therefore the contributions of the counterterms have to respect this invariance. In particular the covariant derivative (2.2) must enter the counterterms as a whole. This imposes some restrictions on possible counterterms: the counterterm required for the 1-irreducible function $\langle\phi'v\phi\rangle$ with $\delta_\Gamma = 1$, $\delta'_\Gamma = 0$, necessarily attains the form $\phi'(v\partial)\phi$ and can appear only in the combination $\phi'\nabla_t\phi$ with the counterterm $\phi'\partial_t\phi$ discussed above. Hence, it is forbidden.
- (4) An additional observation which reduces possible types of counterterms is the generalized Galilean invariance with the time-dependent transformation velocity parameter $\mathbf{w}(t)$:

$$\begin{aligned} \mathbf{v}_w(x) &= \mathbf{v}(x_w) - \mathbf{w}(t), & \Phi_w(x) &= \Phi(x_w), \\ x &= (t, \mathbf{x}), & x_w &= (t, \mathbf{x} + \mathbf{u}(t)), \\ \mathbf{u}(t) &= \int_{-\infty}^t \mathbf{w}(t') dt', \end{aligned} \quad (3.10)$$

where Φ denotes any of the three remaining fields – v', ϕ', ϕ . The crucial idea is that in despite of the fact that the action functional is *not* invariant with respect to such a transformation, it transforms in the identical way with the generating functional of the 1-irreducible Green functions:

$$\begin{aligned} \mathcal{S}(\Phi_w) &= \mathcal{S}(\Phi) + v'\partial_t w, \\ \Gamma(\Phi_w) &= \Gamma(\Phi) + v'\partial_t w. \end{aligned} \quad (3.11)$$

Since the last one can be written in the form

$$\Gamma(\varphi) = \mathcal{S}(\varphi) + \tilde{\Gamma}(\varphi), \quad (3.12)$$

where φ is the set of all fields, $\varphi = \{\mathbf{v}, \mathbf{v}', \phi, \phi'\}$, $\mathcal{S}(\varphi)$ is the action functional, and $\tilde{\Gamma}(\varphi)$ is the sum of all the 1-irreducible loop diagrams that contain all the UV divergences. Expressions (3.11) mean that counterterms appear invariant under the generalized Galilean transformation (3.10).

The above considerations excludes the counterterm $v'\nabla_t v$, invariant with respect to conventional Galilean transformation with a constant vector \mathbf{w} , but not invariant with respect to (3.10). In particular the only possible counterterm for 1-irreducible function $\langle v'v \rangle$ with $\delta_\Gamma = 2$ is $v'\partial^2 v$, and that the 1-irreducible function $\langle v'vv \rangle$ with $\delta_\Gamma = 1$ does not diverge. More detailed discussions of the application of the generalized Galilean transformation can be found in [7, 8, 54, 55].

- (5) From the expressions (3.3) for propagators it follows that the propagators with field ϕ , namely $\langle v'\phi \rangle_0$, $\langle v\phi \rangle_0$, and $\langle \phi\phi \rangle_0$, contain the factor c_0^2 or c_0^4 . Since $d_c^k \neq 0$ and $d_c^\omega \neq 0$, these factors appear as external numerical factors in any diagram involving

TABLE I. Canonical dimensions of the fields and parameters.

F	v'	v	ϕ'	ϕ	θ'	θ	m, μ, Λ	ν_0, ν	c_0, c	g_{10}	g_{20}	$u_0, v_0, w_0, u, v, w, g_1, g_2, \alpha$
d_F^k	$d+1$	-1	$d+2$	-2	d	0	1	-2	-1	y	$4-d$	0
d_F^{ω}	-1	1	-2	2	$1/2$	$-1/2$	0	1	1	0	0	0
d_F	$d-1$	1	$d-2$	2	$d+1$	-1	1	0	1	y	$4-d$	0

these propagators, and its real index of divergence reduces by the corresponding number of unities. In particular, any diagram of the 1-irreducible function with $N_{\phi'} > N_{\phi}$ contains the factor $c_0^{2(N_{\phi'} - N_{\phi})}$. It then follows that the counterterm to the 1-irreducible function $\langle \phi'v \rangle$ with $\delta_{\Gamma} = 3$ necessarily reduces to $c_0^2 \phi'(\partial v)$, while the structures $\phi' \partial^2(\partial v)$, etc. are forbidden. Another consequence is UV finiteness of the 1-irreducible function $\langle \phi'vv \rangle$ with $\delta_{\Gamma} = 2$. Each diagram of this function contains the factor c_0^2 , which forbids the counterterms like $\phi'(\partial v)(\partial v)$, etc., while the remaining structure $c_0^2 \phi'v^2$ is forbidden by the Galilean symmetry.

- (6) An interesting observation refers to the function $\langle v'v' \rangle$: the corresponding index of divergence reads $\delta_{\Gamma} = -d + 4$, therefore it becomes UV divergent in $d = 2, 3$, and 4 and requires a presence of some counterterms. Herewith in physical situation $d = 3$ ($\delta_{\Gamma} = 1$) it is impossible to construct a scalar counterterm containing two vector fields and one derivative¹, so the only possible way is to include UV cut-off Λ in the counterterm. Such counterterms does not affect critical behavior and therefore are not interesting². This means that in this special space dimension $d = 3$ the renormalization group analysis is simplified and does not catch some features, connected with this divergence, and the results of [29, 45] should be treated only as preliminary. The more correct way is to consider our model at $d = 2$ or $d = 4$, which may allow us to find some new scaling regimes, and then return to physical value $d = 3$.

In this work we analyze the model (3.1) at $d = 4$, which requires to consider only one additional divergent function $\langle v'v' \rangle$. For this reason we have modified kernel function $\tilde{D}_{ik}(\mathbf{k})$ [see (3.2)] and introduced the second coupling constant g_{20} and $\varepsilon = 4 - d$, which together with y play a role of an expansion parameter. To explore this model at $d = 2$ one should consider four new divergent functions, namely $\langle v'v' \rangle$ with $\delta_{\Gamma} = 2$, functions $\langle v'v'v' \rangle$

and $\langle v'v'v \rangle$ with $\delta_{\Gamma} = 1$, and function $\langle v'v'v'v' \rangle$ with $\delta_{\Gamma} = 0$, so it is a much more complicated task and a possible problem for the future studies.

Using all these considerations one can check that all the UV divergences in the model (3.1) at $d = 4$ are removed by the counterterms of the form

$$\begin{aligned} v'_i \partial^2 v_i, & \quad v'_i \partial_i \partial_k v_k, & \quad v'_i \partial_i \phi, \\ c_0^2 \phi' \partial_i v_i, & \quad \phi' \partial^2 \phi, & \quad v'v', \end{aligned} \quad (3.13)$$

which are presented in the extended action functional (3.1) with $v_0 > 0$. Now the poles can be eliminated by multiplicative renormalization of the parameters $g_{10}, g_{20}, \nu_0, u_0, v_0, c_0$ and fields ϕ and ϕ' :

$$\begin{aligned} g_{10} &= g_1 \mu^y Z_{g_1}, & u_0 &= u Z_u, & \nu_0 &= \nu Z_{\nu}, \\ g_{20} &= g_2 \mu^{\varepsilon} Z_{g_2}, & v_0 &= v Z_v, & c_0 &= c Z_c. \end{aligned} \quad (3.14)$$

Here μ is the scale-setting parameter (additional free parameter of the renormalized theory) in the minimal subtraction (MS) renormalization scheme, which is always understood in what follows, the parameters g_1, g_2, ν, u, v , and c are renormalized analogs of the bare parameters (without subscript “0”), $Z_i, i \in \{g_1, g_2, u, v, \nu, c\}$, are the renormalization constants, which depend only on the completely dimensionless parameters $g_1, g_2, u, v, \alpha, d, y$ and ε . The fields ϕ and ϕ' are renormalized in the following way:

$$\phi \rightarrow Z_{\phi} \phi, \quad \phi' \rightarrow Z_{\phi'} \phi'. \quad (3.15)$$

The non-local part of the function D_{ik} does not require renormalization, so it is expressed in renormalized parameters using the relation $g_{10} \nu_0^3 = g_1 \nu^3 \mu^y$, see (3.17) below. The parameters m and α from the correlation function (2.7) are not renormalized: $Z_m = Z_{\alpha} = 1$. Due to the absence of renormalization of the term $v' \nabla_t v$ no renormalization of the fields v and v' is needed: $Z_v = Z_{v'} = 1$.

Hence the renormalized action functional has the form

¹ For the same reason the diagram $\langle v'v'v \rangle$ with $\delta_{\Gamma} = 3 - d$ does not diverge at $d = 3$.

² The situation is similar to the well-known ϕ^4 model, in which such counterterms leads to the shift of the parameter $\tau = T - T_c$, which in theory of critical phenomena is an analogue of the mass: $\tau \rightarrow \tau + \Lambda^2$. But we are interesting in the values of critical exponents and are not interesting in the value of critical temperature, therefore we are not interesting in such corrections; moreover,

they are naturally absent in the dimensional regularization and MS scheme. The difference is that in our model there is no local term in (2.8), so this term should appear, with its own constant; see expression (3.2). But if $d = 3$ the new constant g_{20} is not dimensionless and therefore does not affect to the set of β functions and fixed points.

$$\begin{aligned} \mathcal{S}^R(\Phi) = & \frac{1}{2} v'_i D_{ik}^R v'_k + v'_i \left\{ -\nabla_t v_i + Z_1 \nu [\delta_{ik} \partial^2 - \partial_i \partial_k] v_k + Z_2 u \nu \partial_i \partial_k v_k - Z_4 \partial_i \phi \right\} + \\ & + \phi' \left[-\nabla_t \phi + Z_3 v \nu \partial^2 \phi - Z_5 c^2 (\partial_i v_i) \right], \end{aligned} \quad (3.16)$$

where

$$D_{ik}^R = g_1 \mu^y \nu^3 p^{4-d-y} \left\{ P_{ij}(\mathbf{p}) + \alpha Q_{ij}(\mathbf{p}) \right\} + Z_6 g_2 \mu^\varepsilon \nu^3 \delta_{ij}. \quad (3.17)$$

In contrast to the case $d = 3$, there is one more renormalization constant, namely Z_6 , if $d = 4$.

IV. RENORMALIZATION OF THE MODEL

A. Perturbation expansion for the 1-irreducible Green functions

Consider $\Gamma(\varphi)$ – the generating functional of the 1-irreducible Green functions. According to [7], using Leg-

endre transform it may be written in the form

$$\Gamma(\varphi) = \mathcal{S}(\varphi) + \tilde{\Gamma}(\varphi), \quad (4.1)$$

where for the functional arguments we have used the same symbols $\varphi = \{\mathbf{v}, \mathbf{v}', \phi, \phi'\}$ as for the corresponding random fields. Here $\mathcal{S}(\varphi)$ is the action functional (3.1) and $\tilde{\Gamma}(\varphi)$ is the sum of all the 1-irreducible diagrams with loops. Hence in the one-loop approximation expressions for the 1-irreducible Green functions that require UV renormalization, can be formally written in the following fashion:

$$\Gamma_{v'v} = i\omega - (\delta_{ij} p^2 - p_i p_j) Z_1 \nu - p_i p_j Z_2 u \nu + \text{diagram}, \quad (4.2)$$

$$\Gamma_{\phi\phi'} = i\omega - p^2 Z_3 v \nu + \text{diagram}, \quad (4.3)$$

$$\Gamma_{v'\phi} = -i Z_4 p_i + \text{diagram}, \quad (4.4)$$

$$\Gamma_{\phi'v} = -i Z_5 p_i c^2 + \text{diagram} + \text{diagram} + \text{diagram}, \quad (4.5)$$

$$\Gamma_{v'v'} = g_1 \mu^y \nu^3 p^{4-d-y} \left\{ P_{ij}(\mathbf{p}) + \alpha Q_{ij}(\mathbf{p}) \right\} + Z_6 g_2 \mu^\varepsilon \nu^3 \delta_{ij} + \frac{1}{2} \text{diagram}, \quad (4.6)$$

where \mathbf{p} always stands for a corresponding external momentum. A factor 1/2 in front of the diagram in (4.6) denotes a symmetry coefficient of the given graph; all other graphs have symmetry coefficient one.

From the direct comparison of the relations between renormalized parameters it follows that the renormalization constants in (3.14) and (4.2) – (4.6) are related as

$$\begin{aligned} Z_\nu &= Z_1, & Z_{g_1} &= Z_1^{-3}, & Z_c &= (Z_4 Z_5)^{1/2}, \\ Z_\phi &= Z_4, & Z_{\phi'} &= Z_4^{-1}, & Z_v &= Z_3 Z_1^{-1}, \\ Z_u &= Z_2 Z_1^{-1}, & Z_{g_2} &= Z_6 Z_1^{-3}. \end{aligned} \quad (4.7)$$

The renormalization constants are found from the requirement that the Green functions of the renormalized model (3.16), when expressed in renormalized variables, be UV finite. Employing dimensional regularization within MS scheme the renormalization constants can

be calculated and the UV divergences manifests themselves in pole terms in y and $\varepsilon = 4 - d$. In higher loops pole terms in form of general linear combination in $ay + b\varepsilon$ might appear.

B. Calculation of the diagrams. $d_\Gamma = 0$

All diagrams will be calculated at arbitrary space dimension d , and then one should put $d = 4$ in obtained results. Since we are interesting in scaling behavior, it is possible to put $c_0 = 0$ in propagators [see (3.3)] in all situations, when some quantity is not proportional to c_0 . If some quantity is proportional to c_0 , we may put $c_0 = 0$ after we have obtained the needed power of it. In fact this means that we may put $c_0 = 0$ in all denominators, preserving them in numerators. The situation is

similar to calculation of critical exponents in models of critical behavior in massless scheme: we may consider c_0 to play similar rôle as $\tau \propto T - T_c$, the deviation of the temperature from its critical value. In the MS scheme, the renormalization constants do not depend on it and

$$D_1 = (-i)^2 \int \frac{d\omega}{2\pi} \int \frac{d^d \mathbf{k}}{(2\pi)^d} (k_b \delta_{\alpha a} - k_a \delta_{b\alpha}) (k_c \delta_{\beta d} - k_d \delta_{\beta c}) \\ \times \left(P_{ac}(\mathbf{k}) \frac{g_{10} \nu_0^3 k^{4-d-y} + g_{20} \nu_0^3}{|\epsilon_1(k)|^2} + Q_{ac}(\mathbf{k}) (\alpha g_{10} \nu_0^3 k^{4-d-y} + g_{20} \nu_0^3) \left| \frac{\epsilon_3(k)}{R(k)} \right|^2 \right) \\ \times \left(P_{bd}(\mathbf{k}) \frac{g_{10} \nu_0^3 k^{4-d-y} + g_{20} \nu_0^3}{|\epsilon_1(k)|^2} + Q_{bd}(\mathbf{k}) (\alpha g_{10} \nu_0^3 k^{4-d-y} + g_{20} \nu_0^3) \left| \frac{\epsilon_3(k)}{R(k)} \right|^2 \right); \quad (4.8)$$

hereinafter the Greek letters α and β are external (free) indices of the diagram, the Roman letters $a - d$ denote the vector indices of the propagators with the implied summation over repeated indices. Two terms in first line are vertices V_{ijl}^1 (see Fig. 2), the terms in second and third lines are propagators $\langle vv \rangle$, see (3.3) and (3.5). Since $d_\Gamma = 0$ for this diagram, we may put external momenta $\mathbf{p} = 0$.

The calculation of the tensor structure $J_{\alpha\beta}^1$ gives

$$J_{\alpha\beta}^1 = 2(-\delta_{\alpha\beta} k^2 + k_\alpha k_\beta) \times A(k) B(k), \quad (4.9)$$

where $A(k)$ and $B(k)$ are the scalar parts of the propagators in expression (4.8), namely

$$A(k) = \frac{g_{10} \nu_0^3 k^{4-d-y} + g_{20} \nu_0^3}{|\epsilon_1(k)|^2}; \\ B(k) = (\alpha g_{10} \nu_0^3 k^{4-d-y} + g_{20} \nu_0^3) \left| \frac{\epsilon_3(k)}{R(k)} \right|^2. \quad (4.10)$$

The integration over the frequency ω of expression (4.9) gives

$$\int \frac{d\omega}{2\pi} A(k) B(k) = \frac{1}{2k^6 \nu_0^3 u_0(u_0 + 1)}, \quad (4.11)$$

therefore expression (4.8) takes the form

$$D_1 = \int \frac{d^d \mathbf{k}}{(2\pi)^d} \frac{\nu_0^3}{u_0(u_0 + 1)} \frac{1}{k^4} \left(\delta_{\alpha\beta} - \frac{k_\alpha k_\beta}{k^2} \right) \\ \times (g_{10} k^{4-d-y} + g_{20}) (\alpha g_{10} k^{4-d-y} + g_{20}). \quad (4.12)$$

In order to integrate over the vector \mathbf{k} we need to average the expression (4.12) over the angle variables:

$$\int d^d \mathbf{k} f(\mathbf{k}) = S_d \int_m^\infty dk k^{d-1} \langle f(\mathbf{k}) \rangle, \quad (4.13)$$

where $\langle \dots \rangle$ is the averaging over the unit sphere in the d -dimensional space, S_d is its surface area. To average

can be calculated directly at the critical point $\tau = 0$; see [7, 45].

Let us start with the simplest graph for which $d_\Gamma = 0$, entering in expression (4.6). The corresponding analytical expression reads

some function of k over the angles we use the following expressions:

$$\left\langle \frac{k_i k_j}{k^2} \right\rangle = \frac{\delta_{ij}}{d}; \\ \left\langle \frac{k_i k_j k_l k_m}{k^4} \right\rangle = \frac{\delta_{ij} \delta_{lm} + \delta_{il} \delta_{jm} + \delta_{im} \delta_{jl}}{d(d+2)}. \quad (4.14)$$

In particular, first expression of (4.14) means that

$$\int d^d \mathbf{k} \frac{k_i k_s}{k^2} f(k) = \frac{\delta_{is}}{d} \int d^d \mathbf{k} f(k). \quad (4.15)$$

For D_1 this gives

$$D_1 = \frac{\nu_0^3}{u_0(u_0 + 1)} \frac{d-1}{d} \delta_{\alpha\beta} C_d \int d^d k \frac{k^{d-1}}{k^4} (\alpha g_{10}^2 k^{8-2d-2y} \\ + (\alpha + 1) g_{10} g_{20} k^{4-d-y} + g_{20}^2), \quad (4.16)$$

where $C_d = S_d/(2\pi)^d$. After the angular averaging have been performed, we are left with the simple integrals over the modulus k :

$$\int_m^\infty d^d k k^{d-1} \frac{k^{4-d-y}}{k^4} = \frac{m^{-y}}{y}; \\ \int_m^\infty d^d k k^{d-1} \frac{1}{k^4} = \frac{m^{-\varepsilon}}{\varepsilon}, \quad (4.17)$$

where $\varepsilon = 4 - d$. Applying these expressions for (4.16), one obtains that

$$D_1 = \frac{\nu_0^3}{u_0(u_0 + 1)} \frac{d-1}{d} \delta_{\alpha\beta} C_d \\ \times \left(\frac{\alpha g_{10}^2}{2y - \varepsilon} + \frac{(\alpha + 1) g_{10} g_{20}}{y} + \frac{g_{20}^2}{\varepsilon} \right). \quad (4.18)$$

Taking into account symmetry coefficient for this graph, expression (4.6) finally reads

$$+ \nu_0 C_d \frac{(1-u_0)}{2du_0(1+u_0)^2} \times \left(\alpha g_{10} \frac{m^{-y}}{y} + g_{10} \frac{m^{-\varepsilon}}{\varepsilon} \right); \quad (4.60)$$

$$I_{\parallel} = \frac{\nu_0 C_d}{2} (1-d) \frac{u_0^2(d-1) + u_0(d+4) + 1}{d(d+2)(1+u_0)^2} \left(g_{10} \frac{m^{-y}}{y} + g_{10} \frac{m^{-\varepsilon}}{\varepsilon} \right). \quad (4.61)$$

Expressions (4.18), (4.45), (4.49), and (4.59) are final answers for divergent parts of all Green functions, which are needed to renormalize the model. To find renormalization constants it is necessary to put them into expressions (4.2) – (4.6) and to require their UV finiteness (when they are expressed in new renormalized variables), i.e., finiteness at $y \rightarrow 0$ and $\varepsilon \rightarrow 0$. This will be done in the next subsection.

E. Renormalization constants

From expressions (4.2) and (4.59) – (4.61) it follows that renormalization constant Z_1 is connected with the expression I_{\perp} , while renormalization constant Z_2 is connected with expression I_{\parallel} . Moreover, one should not forget all factors like u_0 , v_0 or $g_{1/2,0}$, which are presented in the terms of action functional and are not necessary presented in the results of calculations of diagrams; see, for example, expression (4.19) – not all terms in the expression (4.18) are proportional to the coupling constant g_{20} . Passing to new variables according to a prescription $g_{1,2} \rightarrow g_{1,2} C_d$ one finally obtains

$$\begin{aligned} Z_1 &= 1 + \frac{u^2 d(1-d) - u(2d^2 + 2d - 8) - d(d+3)}{4d(d+2)(1+u)^2} \\ &\quad \times \left(\frac{g_1}{y} + \frac{g_2}{\varepsilon} \right) + \frac{(1-u)}{2du(1+u)^2} \left(\alpha \frac{g_1}{y} + \frac{g_2}{\varepsilon} \right); \\ Z_2 &= 1 + (1-d) \frac{u^2(d-1) + u(d+4) + 1}{2d(d+2)u(1+u)^2} \left(\frac{g_1}{y} + \frac{g_2}{\varepsilon} \right); \\ Z_3 &= 1 - \frac{1}{2dv} \left[\frac{d-1}{1+v} \left(\frac{g_1}{y} + \frac{g_2}{\varepsilon} \right) \right. \\ &\quad \left. + \frac{(u-v)}{u(u+v)^2} \left(\alpha \frac{g_1}{y} + \frac{g_2}{\varepsilon} \right) \right]; \\ Z_4 &= 1 + \frac{(d-1)}{2d(1+u)(1+v)} \left(\frac{g_1}{y} + \frac{g_2}{\varepsilon} \right); \\ Z_6 &= 1 - \frac{(d-1)}{2du(1+u)} \left(\alpha \frac{g_1^2}{g_2(2y-\varepsilon)} + (\alpha+1) \frac{g_1}{y} + \frac{g_2}{\varepsilon} \right). \end{aligned} \quad (4.62)$$

As has been mentioned in Sec. IV C, the renormalization constant Z_5 is trivial. Using relations (4.7) and the binomial relation $(1+x)^{-n} = 1 - nx + \mathcal{O}(x^2)$ to find Z_i^{-n} , one obtains renormalization constants of the fields ϕ and ϕ' and physical parameters of the system:

$$Z_{\nu} = 1 + A \times \left(\frac{g_1}{y} + \frac{g_2}{\varepsilon} \right) + B \times \left(\alpha \frac{g_1}{y} + \frac{g_2}{\varepsilon} \right);$$

$$\begin{aligned} Z_u &= 1 + (C - A) \times \left(\frac{g_1}{y} + \frac{g_2}{\varepsilon} \right) - B \times \left(\alpha \frac{g_1}{y} + \frac{g_2}{\varepsilon} \right); \\ Z_v &= 1 - (A + D) \times \left(\frac{g_1}{y} + \frac{g_2}{\varepsilon} \right) \\ &\quad - (B + E) \times \left(\alpha \frac{g_1}{y} + \frac{g_2}{\varepsilon} \right); \\ Z_c &= 1 + \frac{1}{2} F \times \left(\frac{g_1}{y} + \frac{g_2}{\varepsilon} \right); \\ Z_{\phi} &= 1 + F \times \left(\frac{g_1}{y} + \frac{g_2}{\varepsilon} \right); \\ Z_{\phi'} &= 1 - F \times \left(\frac{g_1}{y} + \frac{g_2}{\varepsilon} \right); \\ Z_{g_1} &= 1 - 3A \times \left(\frac{g_1}{y} + \frac{g_2}{\varepsilon} \right) - 3B \times \left(\alpha \frac{g_1}{y} + \frac{g_2}{\varepsilon} \right); \\ Z_{g_2} &= 1 - 3A \times \left(\frac{g_1}{y} + \frac{g_2}{\varepsilon} \right) - 3B \times \left(\alpha \frac{g_1}{y} + \frac{g_2}{\varepsilon} \right) \\ &\quad - G \times \left[\alpha \frac{g_1^2}{g_2(2y-\varepsilon)} + (1+\alpha) \frac{g_1}{y} + \frac{g_2}{\varepsilon} \right], \end{aligned} \quad (4.63)$$

where $A - F$ are coefficients of renormalization constants $Z_1 - Z_6$:

$$\begin{aligned} A &= \frac{d(1-d)u^2 - 2u(d^2 + d - 4) - d(d+3)}{4d(d+2)(1+u)^2}; \\ B &= \frac{1-u}{2du(1+u)^2}; \\ C &= (1-d) \frac{u^2(d-1) + u(d+4) + 1}{2d(d+2)u(1+u)^2}; \\ D &= \frac{1}{2dv} \frac{d-1}{1+v}; \\ E &= \frac{1}{2dv} \frac{(u-v)}{u(u+v)^2}; \\ F &= \frac{(d-1)}{2d(1+u)(1+v)}; \\ G &= \frac{(d-1)}{2du(1+u)}. \end{aligned} \quad (4.64)$$

Expressions (4.63), (4.64) mean, that we have found all renormalization constants of fields and parameters that are needed to renormalize our model at $d = 4$.

V. RENORMALIZATION GROUP AND CRITICAL SCALING

A. RG equations and RG functions

The relation between initial and renormalized actions functional $\mathcal{S}(\Phi, e_0) = \mathcal{S}^R(Z_\Phi \Phi, e, \mu)$, where e_0 is the complete set of bare parameters and e is the set of their renormalized counterparts, yields to the basic RG differential equation:

$$\{\mathcal{D}_{RG} + N_\phi \gamma_\phi + N_{\phi'} \gamma_{\phi'}\} G^R(e, \mu, \dots) = 0, \quad (5.1)$$

where $G = \langle \Phi \dots \Phi \rangle$ is some correlation function of fields Φ , N_ϕ and $N_{\phi'}$ are numbers of corresponding fields ϕ and ϕ' (requiring some renormalization), entering into correlation function G , ellipsis in expression (5.1) stands for the other arguments of G (spatial and time variables, etc.); \mathcal{D}_{RG} is the operation $\tilde{\mathcal{D}}_\mu$ expressed in the renormalized variables and $\tilde{\mathcal{D}}_\mu$ is the differential operation $\mu \partial_\mu$ for

fixed e_0 :

$$\mathcal{D}_{RG} = \mathcal{D}_\mu + \beta_{g_1} \partial_{g_1} + \beta_{g_2} \partial_{g_2} + \beta_u \partial_u + \beta_v \partial_v - \gamma_\nu \mathcal{D}_\nu - \gamma_c \mathcal{D}_c. \quad (5.2)$$

Here we have denoted $\mathcal{D}_x \equiv x \partial_x$ for any variable x . The anomalous dimension γ_F of a certain quantity F (a field or a parameter) is defined as

$$\gamma_F = Z_F^{-1} \tilde{\mathcal{D}}_\mu Z_F = \tilde{\mathcal{D}}_\mu \ln Z_F, \quad (5.3)$$

and the β functions for the four dimensionless coupling constants g_1 , g_2 , u and v , which express the flows of parameters under the RG transformation, are $\beta_g = \tilde{\mathcal{D}}_\mu g$. Together with (3.14) this yields

$$\begin{aligned} \beta_{g_1} &= g_1 (-y - \gamma_{g_1}), & \beta_{g_2} &= g_2 (-\varepsilon - \gamma_{g_2}), \\ \beta_u &= -u \gamma_u, & \beta_v &= -v \gamma_v. \end{aligned} \quad (5.4)$$

From the definitions and explicit expressions (4.63), (4.64) one finds in the one-loop approximation (i.e., with corrections of orders g_1^2 , g_2^2 , $g_1 g_2$ and higher) at $d = 4$:

$$\gamma_\nu = g_1 \frac{(3u^2 + 8u + 7)}{24(1+u)^2} + \alpha g_1 \frac{(u-1)}{8u(1+u)^2} + g_2 \frac{(3u^3 + 8u^2 + 10u - 3)}{24u(1+u)^2}; \quad (5.5)$$

$$\gamma_u = \frac{1-u}{48u(1+u)^2} \left[g_1(6u^2 + 13u + 3) + 6\alpha g_1 + g_2(6u^2 + 13u + 9) \right]; \quad (5.6)$$

$$\begin{aligned} \gamma_v &= \frac{g_1}{24} \left[-\frac{7+8u+3u^2}{(1+u)^2} + \frac{9}{v(1+v)} \right] - \alpha g_1 \frac{(v-1)}{8u(1+u)^2 v(u+v)^2} \left[u^3 + 2u^2(1+v) - v(1+v) + u(1-v+v^2) \right] \\ &+ \frac{g_2}{24} \left[\frac{3(1-u)}{u(1+u)^2} - \frac{7+8u+3u^2}{(1+u)^2} + \frac{3(u-v)}{uv(u+v)^2} + \frac{9}{v(1+v)} \right]; \end{aligned} \quad (5.7)$$

$$\gamma_c = -\frac{3}{16(1+u)(1+v)}(g_1 + g_2); \quad (5.8)$$

$$\gamma_\phi = -\frac{3}{8(1+u)(1+v)}(g_1 + g_2); \quad (5.9)$$

$$\gamma_{\phi'} = \frac{3}{8(1+u)(1+v)}(g_1 + g_2); \quad (5.10)$$

$$\gamma_{g_1} = -g_1 \frac{(3u^2 + 8u + 7)}{8(1+u)^2} - \alpha g_1 \frac{3(u-1)}{8u(1+u)^2} - g_2 \frac{(3u^3 + 8u^2 + 10u - 3)}{8u(1+u)^2}; \quad (5.11)$$

$$\gamma_{g_2} = \frac{1}{8u(1+u)^2} \left[-g_1(3u^3 + 8u^2 + 4u - 3) - g_2(3u^3 + 8u^2 + 7u - 6) + 3 \frac{\alpha g_1}{g_2} [(1+u)g_1 + 2g_2] \right]. \quad (5.12)$$

This means that from expressions (5.4) and (5.5) – (5.12) all functions, which enters into differential operator (5.2), are known, and therefore it can be estimated how this differential operator acts on different Green functions. We do not include the dimensionless parameter α into the list of coupling constants, because it is not renormalized ($Z_\alpha = 1$) and the corresponding function β_α vanishes identically. Thus there is no restriction to the value of α from RG equations, and it remains a free

parameter of the theory.

B. RG functions and IR attractive fixed points

One of the basic RG statements is that the large scale behavior with respect to spatial and time scales is governed by the IR attractive stable fixed points $g^* \equiv \{g_1^*, g_2^*, u^*, v^*\}$, which coordinates are found from the fol-

lowing condition [7, 9, 10]:

$$\beta_{g_1}(g^*) = \beta_{g_2}(g^*) = \beta_u(g^*) = \beta_v(g^*) = 0. \quad (5.13)$$

The idea is to consider a set of invariant charges $\bar{g}_i = \bar{g}_i(s, g)$ with the initial data $\bar{g}_i|_{s=1} = g_i$. The parameter $s = k/\mu$ is a scaling parameter and IR asymptotic behavior (i.e., behavior as $r \rightarrow \infty$) corresponds to the limit $s \rightarrow 0$. The evolution of invariant charges is described by the set of flow equations

$$\mathcal{D}_s \bar{g}_i = \beta_i(\bar{g}_j), \quad (5.14)$$

whose solution as $s \rightarrow 0$ behaves approximately like

$$\bar{g}_i(s, g^*) \cong g^* + \text{const} \times s^{\omega_i}, \quad (5.15)$$

where $\{\omega_i\}$ is the set of eigenvalues of the matrix

$$\Omega_{ij} = \partial \beta_i / \partial g_j|_{g=g^*}. \quad (5.16)$$

The existence of IR stable solutions of the RG equations leads to the existence of the scaling behavior of Green functions. From (5.15) it follows that the type of a fixed point is determined by the matrix (5.16): for the IR stable fixed points the matrix Ω has to be positive definite, i.e., the real parts of all its eigenvalues must be positive.

In contrast to the case three dimensional case, where the analysis of the expressions like (5.4) and (5.5) – (5.12) has shown that in the physical range of parameters $g_1, g_2, u, v, \alpha > 0$ there exist only one nontrivial IR attractive fixed point [29, 45], at $d = 4$ situation is more intriguing: a direct analysis of the system of equations (5.13) reveals the existence of three IR stable fixed points: trivial free (Gaussian) fixed point FPI and two nontrivial fixed points FPII and FPIII.

The free fixed point FPI, for which all interactions are irrelevant and no scaling and universality is expected, has coordinates

$$g_1^* = 0, \quad g_2^* = 0, \quad (5.17)$$

whereas coordinates for charges u^* and v^* can not be determined. The corresponding eigenvalues of the matrix Ω_{ij} are

$$\lambda_1 = 0, \quad \lambda_2 = 0, \quad \lambda_3 = -\varepsilon, \quad \lambda_4 = -y. \quad (5.18)$$

Though trivial, this point is necessary for the correct use of perturbative renormalization group. From (5.18) it follows that it is IR attractive for negative values of y and ε . Two zero eigenvalues means that in four-dimensional space of coupling constants $\{g_1, g_2, u, v\}$ this point is a “point” only in two dimensions $\{g_1, g_2\}$; e.g., in the picture of RG flux in the plane $\{g_1, u\}$ FPI is not an attractive point, but it is an attractive line, with zero velocity along direction $u = 0$.

For second fixed point FPII only $g_1^* = 0$ while $g_2^* \neq 0$, therefore this scaling regime is called “local” [see (3.2)]. Its coordinates are

$$g_1^* = 0, \quad g_2^* = \frac{8\varepsilon}{3}, \quad u^* = 1, \quad v^* = 1. \quad (5.19)$$

The eigenvalues of the matrix Ω_{ij} are

$$\lambda_1 = \frac{7\varepsilon}{18}, \quad \lambda_2 = \frac{5\varepsilon}{6}, \quad \lambda_3 = \varepsilon, \quad \lambda_4 = \frac{3\varepsilon - 2y}{2}, \quad (5.20)$$

from what follows that this point is IR attractive in the region given by the inequalities $\varepsilon > 0$ and $y < 3\varepsilon/2$.

For the last fixed point, FPIII, both non-local and local parts of the random force are relevant:

$$g_1^* = \frac{16y(2y - 3\varepsilon)}{9[y(2 + \alpha) - 3\varepsilon]}, \quad g_2^* = \frac{16\alpha y^2}{9[y(2 + \alpha) - 3\varepsilon]}, \\ u^* = 1, \quad v^* = 1. \quad (5.21)$$

The required eigenvalues of the matrix Ω_{ij} are

$$\lambda_1 = \frac{y[2y(10\alpha + 11) - 3\varepsilon(3\alpha + 11)]}{54[y(2 + \alpha) - 3\varepsilon]}, \\ \lambda_2 = \frac{y[2y(2\alpha + 3) - \varepsilon(\alpha + 9)]}{6[y(\alpha + 2) - 3\varepsilon]}, \\ \lambda_{3,4} = \frac{A \pm \sqrt{B}}{C}, \quad (5.22)$$

where the constants A, B , and C are given by the following expressions:

$$A = -27\varepsilon^3 + 9(9 + \alpha)\varepsilon^2 y - 9(8 + 3\alpha)\varepsilon y^2 \\ + 2y^3(\alpha^2 + 7\alpha + 10); \\ B = [-3\varepsilon + (2 + \alpha)y]^2 [81\varepsilon^4 - 54\varepsilon^3 y - 9(3 + 20\alpha)\varepsilon^2 y^2 \\ + 12(1 + 17\alpha + 3\alpha^2)\varepsilon y^3 - 4(-1 + 14\alpha + 5\alpha^2)y^4]; \\ C = 6[-3\varepsilon + (2 + \alpha)y]^2. \quad (5.23)$$

Taking into account that in physical range charges g_1 and g_2 must be positive, it follows from the explicit form of the eigenvalues $\lambda_1 \dots \lambda_4$ that the point FPIII is stable when $y > 0$ and $y > 3\varepsilon/2$.

To complete analysis of fixed points' structure also infinite fixed points values $u \rightarrow \infty$ and $v \rightarrow \infty$ have to be considered. Since u may be interpreted as longitudinal viscosity, from physical point of view this case corresponds to the limit $c \rightarrow \infty$. As it should be seen from these reasoning, velocity fields \mathbf{v} and \mathbf{v}' become purely transverse and scalar fields ϕ and ϕ' effectively decouple from them; see explicit expressions for propagators (3.3). Introducing a new variable $f = 1/u$ with its β function

$$\beta_f = \tilde{\mathcal{D}}_\mu f = -f^2 \beta_u \quad (5.24)$$

one obtains the following set of β functions at $f = 0$:

$$\beta_{g_1} = \frac{1}{8}g_1(3g_1 + 3g_2 - 8y); \\ \beta_{g_2} = \frac{1}{8}g_2(3g_1 + 3g_2 - 8\varepsilon); \\ \beta_v = \frac{1}{8}(g_1 + g_2)\frac{v^2 + v - 3}{v + 1}. \quad (5.25)$$

From (5.24) and (5.25) it follows that there are two non-trivial fixed points for $f_* = 0$:

$$g_1 = 0, \quad g_2 = 8\varepsilon/3, \quad v = \frac{1}{2}(-1 + \sqrt{13}); \quad (5.26)$$

$$g_1 = 8y/3, \quad g_2 = 0, \quad v = \frac{1}{2}(-1 + \sqrt{13}). \quad (5.27)$$

But for both of them at any values of y and ε two of four eigenvalues have different signs, namely:

$$\lambda_1 = -y/3, \quad \lambda_2 = \frac{2(13 + \sqrt{13})}{3(1 + \sqrt{13})^2}y \quad (5.28)$$

for fixed point (5.26), whereas

$$\lambda_1 = -\varepsilon/3, \quad \lambda_2 = \varepsilon \quad (5.29)$$

for fixed point (5.27).

That means both fixed points (5.26) and (5.27) are unstable (i.e., a saddle-like points), what is in agreement with the idea that leading-order correction in the Mach number to the incompressible scaling regime destroys its stability [42–44].

In order to study the limit $v \rightarrow \infty$ let us pass to a new variable $t = 1/v$ with β function

$$\beta_t = \tilde{\mathcal{D}}_\mu t = -t^2 \beta_v. \quad (5.30)$$

As expected for $t = 0$ one obtains $\beta_t = 0$. Since β functions of other coupling constants g_1 , g_2 and u are independent of v , at $t = 0$ we recognize formerly obtained fixed points FPII and FPIII. This means that to investigate the IR attraction of these two points one should check only the derivative $\partial\beta_t/\partial t$ at fixed point $\{g^*, t = 0\}$:

$$\begin{aligned} \lambda_t &= -\varepsilon/2 \quad \text{for FPII;} \\ \lambda_t &= -y/3 \quad \text{for FPIII.} \end{aligned} \quad (5.31)$$

From comparison with (5.20) and (5.22) it follows that at the limit $v \rightarrow \infty$ these two fixed points are saddle type points as well.

As for the three-dimensional case $d = 3$ (see [29]), both IR attractive fixed points FPII and FPIII are valid for all $\alpha > 0$, have finite limits at $\alpha \rightarrow \infty$, but become unstable at $\alpha \rightarrow \infty$, i.e., in the case of purely potential random force. To study this limit, it is reasonable to define new coupling constants $g'_{1,2} = g_{1,2}\alpha$, which are finite as $\alpha \rightarrow \infty$. Hence two new β functions are

$$\beta_{g'_{1,2}} = \tilde{\mathcal{D}}_\mu g'_{1,2} = \alpha \beta_{g_{1,2}}, \quad (5.32)$$

and full set of β functions read

$$\begin{aligned} \beta_{g'_1} &= g'_1 \left(-y + g'_1 \frac{3(u-1)}{8u(u+1)^2} \right); \\ \beta_{g'_2} &= g'_2 \left(-\varepsilon - \frac{1}{8u(u+1)^2} \frac{3g'_1}{g'_2} [(1+u)g'_1 + 2g'_2] \right); \end{aligned}$$

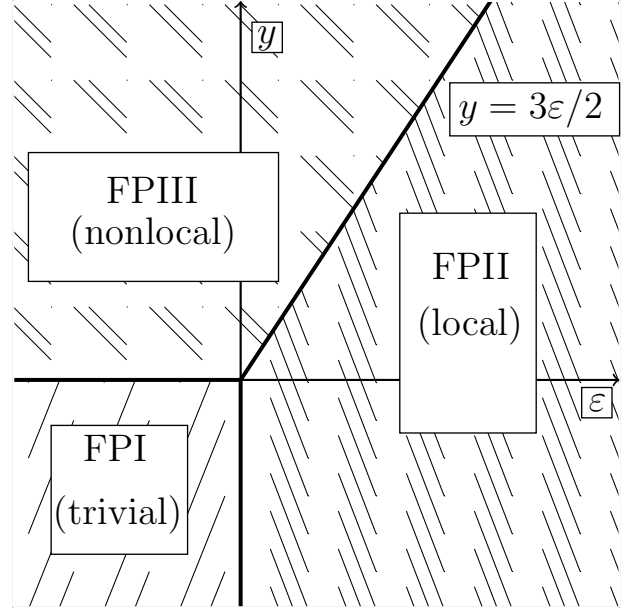


FIG. 3. Domains of IR stability of the fixed points in the model (3.1)

$$\begin{aligned} \beta_u &= g'_1 \frac{(u-1)}{8(u+1)^2}; \\ \beta_v &= g'_1 \frac{(v-1)}{8u(u+1)^2(u+v)^2} \\ &\quad \times [u^3 + 2u^2(1+v) - v(v+1) + u(1-v+v^2)]. \end{aligned} \quad (5.33)$$

Solution to this system is Gaussian point FPI:

$$g_1^* = 0, \quad g_2^* = 0, \quad (5.34)$$

with u^* and v^* undetermined. The corresponding eigenvalues of the matrix Ω_{ij}

$$\lambda_1 = 0, \quad \lambda_2 = 0, \quad \lambda_3 = -\varepsilon, \quad \lambda_4 = -y, \quad (5.35)$$

i.e., this point is IR attractive when $y < 0$ and $\varepsilon < 0$.

Thus in the one-loop level two nontrivial fixed points FPII and FPIII cease to exist at $\alpha \rightarrow \infty$, and the only possible one is free Gaussian point (5.34). However, a new fixed point can appear at the two-loop level.

The general pattern of the stability of three fixed points in the $y - \varepsilon$ plane is shown in Fig. 3. The straight lines $y < 0$, $\varepsilon = 0$; $y = 0$, $\varepsilon < 0$; and $y = 3\varepsilon/2$, $\varepsilon > 0$ corresponds to the boundaries of domains, which has neither gaps nor overlaps between them. The crossover between two nontrivial fixed points happens along the line $y = 3\varepsilon/2$, which is in accordance with [28].

This fact implies that depending of the values y and ε correlation functions of the model (3.1) in the IR region exhibit different types of scaling behavior. The corresponding critical dimensions $\Delta[F] \equiv \Delta_F$ for all basic fields and parameters can be calculated as series in y and ε ; see next subsection.

C. IR attractive fixed points and critical dimensions

In the leading order of the IR asymptotic behavior the Green functions satisfy the RG equation (5.1) with the substitution $g \rightarrow g_*$ for the full set of the couplings [7, 10]. This gives

$$\left\{ \mathcal{D}_\mu - \gamma_\nu^* \mathcal{D}_\nu - \gamma_c^* \mathcal{D}_c + \sum_\Phi N_\Phi \gamma_\Phi^* \right\} G^R = 0. \quad (5.36)$$

Here γ_F^* is the value of the anomalous dimension at fixed-point; the summation over all types of the fields Φ is implied. Equations of this type describe the scaling with dilatation of the variables whose derivatives enter the differential operator.

From (5.5) – (5.12) one obtains that in the one-loop approximation expressions for the anomalous dimensions γ_F^* for non-local fixed point FPIII coincide with the case of $d = 3$:

$$\begin{aligned} \gamma_\nu^* &= y/3, \quad \gamma_\phi^* = -\gamma_{\phi'}^* = -y/6 + \mathcal{O}(y^2), \\ \gamma_c^* &= -y/12 + \mathcal{O}(y^2). \end{aligned} \quad (5.37)$$

For local fixed point FPII one obtains

$$\begin{aligned} \gamma_\nu^* &= \varepsilon/3, \quad \gamma_\phi^* = -\gamma_{\phi'}^* = -\varepsilon/4 + \mathcal{O}(\varepsilon^2), \\ \gamma_c^* &= -\varepsilon/8 + \mathcal{O}(\varepsilon^2). \end{aligned} \quad (5.38)$$

Canonical scale invariance is expressed by two relations

$$\left[\sum_\sigma d_\sigma^k \mathcal{D}_\sigma - d_G^k \right] G^R = 0, \quad \left[\sum_\sigma d_\sigma^\omega \mathcal{D}_\sigma - d_G^\omega \right] G^R = 0, \quad (5.39)$$

where σ is the full set of all arguments of G^R , d^k and d^ω are canonical dimensions. In order to derive the scaling with fixed “IR irrelevant” parameters μ and ν one has to combine equations (5.36) and (5.39) such that the derivatives with respect to these parameters be eliminated; see [7, 8]. This gives an equation of critical IR scaling for the model:

$$\left\{ -\mathcal{D}_\mathbf{x} + \Delta_t \mathcal{D}_t + \Delta_c \mathcal{D}_c + \Delta_m \mathcal{D}_m - \sum_\Phi N_\Phi \Delta_\Phi \right\} G^R = 0 \quad (5.40)$$

with

$$\Delta_F = d_F^k + \Delta_\omega d_F^\omega + \gamma_F^*, \quad \Delta_\omega = -\Delta_t = 2 - \gamma_\nu^*. \quad (5.41)$$

Here Δ_F is the critical dimension of the quantity F , while Δ_t and Δ_ω are the critical dimensions of the time and the frequency.

From Table I and expressions (5.37) and (5.38) one obtains that for fixed point FPIII they coincide with the case $d = 3$, namely

$$\Delta_v = 1 - y/3, \quad \Delta_{v'} = d - \Delta_v,$$

$$\begin{aligned} \Delta_\omega &= 2 - y/3, & \Delta_m &= 1, \\ \Delta_\phi &= d - \Delta_{\phi'} = 2 - 5y/6, & \Delta_c &= 1 - 5y/12. \end{aligned} \quad (5.42)$$

For local fixed point FPII they are following:

$$\begin{aligned} \Delta_v &= 1 - \varepsilon/2, & \Delta_{v'} &= d - \Delta_v, \\ \Delta_\omega &= 2 - \varepsilon/2, & \Delta_m &= 1, \\ \Delta_\phi &= d - \Delta_{\phi'} = 2 - 5\varepsilon/4, & \Delta_c &= 1 - 5\varepsilon/8. \end{aligned} \quad (5.43)$$

This means that depending on the values y and ε correlation functions exhibit different types of scaling behavior in the IR region (local regime FPII or non-local regime FPIII), with different anomalous and critical dimensions.

VI. ADVECTION OF PASSIVE SCALAR FIELDS

The analysis of the passive advection bears a close resemblance to the case $d = 3$ (see [29]), so we will restrict ourselves to the main points.

A. Field theoretic formulation of the model

Consider passive advection of a scalar density field $\theta(x) \equiv \theta(t, \mathbf{x})$, which satisfies the following stochastic differential equation

$$\partial_t \theta + \partial_i (v_i \theta) = \kappa_0 \partial^2 \theta + f_\theta. \quad (6.1)$$

Another formulation of problem, where in the left hand side $\partial_i (v_i \theta) \rightarrow (v_i \partial_i) \theta$, corresponds to a passive advection of a tracer field (e.g., temperature, concentration of the impurity particles, etc.); see [52].

As usual, $\partial_t \equiv \partial/\partial t$, $\partial_i \equiv \partial/\partial x_i$; κ_0 is the molecular diffusivity coefficient, $\partial^2 = \partial_i \partial_i$ is the Laplace operator, $\mathbf{v}(x)$ is the velocity field, which obeys equation (2.1), and $f_\theta \equiv f_\theta(x)$ is a Gaussian noise with zero mean and given covariance

$$\langle f_\theta(x) f_\theta(x') \rangle = \delta(t - t') C(\mathbf{r}/L), \quad \mathbf{r} = \mathbf{x} - \mathbf{x}' \quad (6.2)$$

with some function $C(\mathbf{r}/L)$, which is finite as $(\mathbf{r}/L) \rightarrow 0$ and rapidly decaying when $(\mathbf{r}/L) \rightarrow \infty$. Expression (6.2) brings about another large (integral) scale L , related to the noise variable f_θ , but henceforth we will not distinguish it and its analog $L = m^{-1}$ in the correlation function of the stirring force (2.7). The noise is necessary to account for the effects of initial and/or boundary conditions.

In the absence of the noise, equation (6.1) has the form of a continuity equation (conservation law); θ being the density of a corresponding conserved quantity. If the function in (6.2) is chosen such that its Fourier transform $C(\mathbf{k})$ vanishes at $\mathbf{k} = 0$, the fields θ or θ' remain to be conserved in statistical sense in the presence of the external stirring.

The advection of such fields in the case of Kraichnan's rapid-change velocity ensemble were thoroughly studied [32–39]; the case of Gaussian velocity statistics with finite correlation time was studied in [40, 41].

If velocity \mathbf{v} obeys stochastic Navier-Stokes equation (2.1), the problem (6.1), (6.2) is tantamount to the field theoretic model of the full set of fields $\Phi \equiv \{\theta', \theta, v', v, \phi', \phi\}$ and action functional

$$\mathcal{S}_\Phi(\Phi) = \mathcal{S}_\theta(\theta', \theta, v) + \mathcal{S}(v', v, \phi', \phi), \quad (6.3)$$

where

$$\mathcal{S}_\theta(\theta', \theta, v) = \frac{1}{2} \theta' D_f \theta' + \theta' \{ -\partial_t \theta - \partial_i (v_i \theta) + \kappa_0 \partial^2 \theta \} \quad (6.4)$$

is the De Dominicis–Janssen action for the stochastic problem (6.1), (6.2) at fixed \mathbf{v} , while the second term is given by (3.1) and represents the velocity statistics; D_f is the correlation function (6.2), all the required integrations and summations over the vector indices are assumed.

In addition to (3.3), the diagrammatic technique in the full problem involves new vertex $V_i^3 = -\theta' \partial_i (v_i \theta)$ (depicted on Fig. 4), which in the momentum representation is equivalent to

$$V_j(\mathbf{k}) = i k_j, \quad (6.5)$$

where \mathbf{k} being the momentum carried by the field θ' , and two new propagators

$$\begin{aligned} \langle \theta \theta' \rangle_0 &= \langle \theta' \theta \rangle_0^* = \frac{1}{-i\omega + \kappa_0 k^2}, \\ \langle \theta \theta \rangle_0 &= \frac{C(\mathbf{k})}{\omega^2 + \kappa_0^2 k^4}. \end{aligned} \quad (6.6)$$

From now on a double solid line without a slash denotes the field θ , and double solid line with a slash corresponds the field θ' ; see Fig. 5

B. Renormalization of the model

Canonical dimensions of the new fields and parameters of the full model (6.3) are given in Table I, where we have introduced a new dimensionless parameter $w_0 = \kappa_0 / \nu_0$ with ν_0 from (2.1).

The expression (3.8) for the formal index of UV divergence remains valid, but the summation should now run

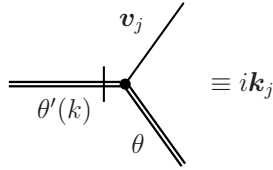


FIG. 4. Graphical representation of the interaction vertex V_j^3 .

over the full set of six fields $\Phi \equiv \{\theta', \theta, v', v, \phi', \phi\}$. The rules (1) – (6) from Sec. IIIB should be generalized and augmented as follows:

- (7) All the 1-irreducible Green functions without the response field θ' vanish identically and require no counterterms.
- (8) Using integration by parts the derivative at the vertex $-\theta' \partial_i (v_i \theta)$ can be moved onto the field θ' , therefore expression (3.9) should be modified as

$$\delta'_\Gamma = \delta_\Gamma - N_\phi - N_{\theta'}. \quad (6.7)$$

Since the field θ' can enter the counterterms only in the form of spatial derivatives, the counterterm $\theta' \partial_i \theta$ to the 1-irreducible Green function $\langle \theta' \theta \rangle$ with $\delta_\Gamma = 2$, $\delta'_\Gamma = 1$ is forbidden. Also this requires that the counterterms to the 1-irreducible function $\langle \theta' v \theta \rangle$ with $\delta_\Gamma = 1$, $\delta'_\Gamma = 0$ necessarily reduce to the form $\theta' \partial_i (v_i \theta)$. Galilean symmetry wherein allows them enter the counterterms only in the form of invariant combination $\theta' \nabla_i \theta$. Hence, they are also forbidden.

- (9) As consequence of the linearity of the original stochastic equation (6.1) with respect to the field θ one obtains that for any 1-irreducible function $N_{\theta'} - N_\theta = 2N_0$, where $N_0 \geq 0$ is the total number of bare propagators $\langle \theta \theta \rangle_0$ entering the diagram. This fact is very important for the renormalizability of the model: without the restriction $N_\theta \leq N_{\theta'}$, the infinity number of superficially divergent 1-irreducible functions $\langle \theta' \theta \dots \theta \rangle$ would proliferate, and hence the lack of renormalizability would follow.

From these rules we finally conclude that superficial divergences can be present only in 1-irreducible Green function $\langle \theta' \theta \rangle$ with the only counterterm $\theta' \partial^2 \theta$. It is naturally reproduced as multiplicative renormalization of the diffusion coefficient, $\kappa_0 = \kappa Z_\kappa$. No renormalization of the fields θ' and θ is needed: $Z_{\theta'} = Z_\theta = 1$. The renormalized analog of the action functional (6.3) has the form

$$\mathcal{S}_\Phi^R(\Phi) = \mathcal{S}_\theta^R(\theta', \theta, v) + \mathcal{S}^R(v', v, \phi', \phi), \quad (6.8)$$

where \mathcal{S}^R is the action (3.16),


$$\begin{aligned} \mathcal{S}_\theta^R(\theta', \theta, v) &= \frac{1}{2} \theta' D_f \theta' + \theta' \{ -\partial_t \theta - \partial_i (v_i \theta) + \kappa Z_\kappa \partial^2 \theta \}; \\ D_f &\text{ here stands for the covariance of stochastic force (6.2).} \end{aligned} \quad (6.9)$$

$$\theta \text{ (double solid line) } \equiv \theta' \text{ (double solid line with slash)} \quad \theta \text{ (double solid line) } \equiv \theta \text{ (double solid line)}$$

FIG. 5. Graphical representation of the bare propagators $\langle \theta \theta' \rangle_0$ and $\langle \theta \theta \rangle_0$.

C. Calculation of the diagram, fixed points and critical dimensions

The one-loop approximation for the 1-irreducible response function $\langle \theta' \theta \rangle$ can be formally written as

$$\Gamma_{\theta' \theta} = +i\omega - \kappa_0 p^2 + \text{diagram}, \quad (6.10)$$


where, as earlier in expressions (4.2) – (4.6), \mathbf{p} represents an external momentum; single solid line denotes the bare propagator $\langle vv \rangle_0$ from (3.3), double solid line with a slash denotes the bare propagator $\langle \theta \theta' \rangle_0$ from (6.6), the slashed end corresponds to the field θ' . The dots with three attached fields θ' , θ and v denote the vertex (6.5).

The constant Z_κ is to be found from the requirement of UV finiteness of 1-irreducible Green function $\langle \theta' \theta \rangle$. Like for the original Navier-Stokes model, the divergent part of the considered Feynman diagram is independent on $c_0 \sim c$ and therefore can be calculated directly at $c = 0$; see the discussion in Sec. IV B.

The analytical expression for the diagram D_8 , entering into expression (6.10), is

$$D_8 = \int \frac{d\omega}{2\pi} \int \frac{d^d \mathbf{k}}{(2\pi)^d} V_a(\mathbf{p}) V_c(\mathbf{p} + \mathbf{k}) \frac{1}{-i\omega + w_0 \nu(\mathbf{p} + \mathbf{k})^2} \times [P_{ac}(\mathbf{k}) \times A(k) + Q_{ac}(\mathbf{k}) \times B(k)], \quad (6.11)$$

where $V_a(\mathbf{p}) = ip_a$ and $V_c(\mathbf{p} + \mathbf{k}) = i(p + k)_c$ are two vertices of the type (6.5); scalar coefficients $A(k)$ and $B(k)$ of the propagator $\langle vv \rangle$ are defined in (4.10).

Since in the leading-order approximation the renormalization constant Z_κ in the bare term of (6.10) is taken only in the first order in coupling constants g_1 and g_2 ,

during the actual calculation all other renormalization constants in the diagram D_8 , entering, for example, into functions $A(k)$ and $B(k)$, should be replaced with unities.

Integration over the frequency gives

$$\int \frac{d\omega}{2\pi} \frac{A(k)}{-i\omega + w_0 \nu(\mathbf{p} + \mathbf{k})^2} = \frac{1}{2\nu^2 k^2 [k^2 + w_0(\mathbf{p} + \mathbf{k})^2]},$$

$$\int \frac{d\omega}{2\pi} \frac{B(k)}{-i\omega + w_0 \nu(\mathbf{p} + \mathbf{k})^2} = \frac{1}{2\nu^2 u k^2 [u k^2 + w_0(\mathbf{p} + \mathbf{k})^2]}. \quad (6.12)$$

Divide expression (6.11) into two parts:

$$\tilde{I}_1 = - \int \frac{d^d \mathbf{k}}{(2\pi)^d} p_a(p + k)_c \times \frac{P_{ac}(\mathbf{k})}{2\nu^2 k^2 [k^2 + w_0(\mathbf{p} + \mathbf{k})^2]},$$

$$\tilde{I}_2 = - \int \frac{d^d \mathbf{k}}{(2\pi)^d} p_a(p + k)_c \times \frac{Q_{ac}(\mathbf{k})}{2\nu^2 u k^2 [u k^2 + w_0(\mathbf{p} + \mathbf{k})^2]}. \quad (6.13)$$

We are interesting in the term, proportional to p^2 , therefore in calculation of \tilde{I}_1 one may immediately put $\mathbf{p} = 0$ and, using expressions (4.14) and (4.17) and notation $g_i C_d \rightarrow g_i$, gets

$$\tilde{I}_1 = -p^2 \frac{\nu}{2(1 + w_0)} \frac{d-1}{d} \left(g_1 \frac{m^{-y}}{y} + g_2 \frac{m^{-\varepsilon}}{\varepsilon} \right). \quad (6.14)$$

In calculation of \tilde{I}_2 the second expression of (6.12) has to be expanded up to the linear term in \mathbf{p} ; in the end we get

$$\tilde{I}_2 = -p^2 \frac{\nu}{2u(u + w_0)} \frac{1}{d} \frac{u - w_0}{u + w_0} \left(\alpha g_1 \frac{m^{-y}}{y} + g_2 \frac{m^{-\varepsilon}}{\varepsilon} \right). \quad (6.15)$$

Finally, collecting \tilde{I}_1 and \tilde{I}_2 yields

$$D_8 = -p^2 \frac{\nu}{2d} \left[\left(\frac{d-1}{1+w_0} + \frac{\alpha}{u(u+w_0)} - 2 \frac{\alpha w_0}{u(u+w_0)^2} \right) \left(\frac{\mu}{m} \right)^y \frac{g_1}{y} - \left(\frac{d-1}{1+w_0} + \frac{1}{u(u+w_0)} - 2 \frac{w_0}{u(u+w_0)^2} \right) \left(\frac{\mu}{m} \right)^\varepsilon \frac{g_2}{\varepsilon} \right]. \quad (6.16)$$

Then the renormalization constant Z_κ [see (6.10)] should be chosen as

$$Z_\kappa = 1 - \frac{1}{2dw} \left[\frac{d-1}{1+w} + \frac{\alpha(u-w)}{u(u+w)^2} \right] \frac{g_1}{y} - \frac{1}{2dw} \left[\frac{d-1}{1+w} + \frac{u-w}{u(u+w)^2} \right] \frac{g_2}{\varepsilon}, \quad (6.17)$$

while the corresponding anomalous dimension is

$$\gamma_\kappa = \frac{1}{2dw} \left[\frac{d-1}{1+w} + \frac{\alpha(u-w)}{u(u+w)^2} \right] g_1 + \frac{1}{2dw} \left[\frac{d-1}{1+w} + \frac{u-w}{u(u+w)^2} \right] g_2, \quad (6.18)$$

with the possible corrections coming from higher orders terms.

The function $\beta_w = \tilde{\mathcal{D}}_\mu w$ for the new dimensionless parameter w has the form

$$\beta_w = -w\gamma_w = w(\gamma_\nu - \gamma_\kappa), \quad (6.19)$$

see Sec. V A. Now the coordinates $\{g^*\}$ of fixed points FPII and FPIII [see (5.19) and (5.21)] are substituted into the equation $\beta_w = 0$ at $d = 4$. To this end we rewrite expression for $\gamma_\nu - \gamma_\kappa$ at $u = 1$:

$$\gamma_\nu - \gamma_\kappa|_{u=1} = \frac{w-1}{16w(w+1)^2} [g_1(6 + 2\alpha + 9w + 3w^2)]$$

$$+g_2(8+9w+3w^2)] . \quad (6.20)$$

From equation (6.20) it is clear that the only positive solution for both FPII and FPIII is

$$w^* = 1. \quad (6.21)$$

Functions (5.4) do not depend on w , therefore new eigenvalue of the matrix (5.16), corresponding to this parameter, coincides with the diagonal element $\partial\beta_w/\partial w$ at point $\{g\} = \{g^*\}$:

$$\begin{aligned} \lambda_w &= \frac{5\varepsilon}{6} > 0 \quad \text{for FPII;} \\ \lambda_w &= \frac{2y}{3} + \frac{4\alpha y(y-\varepsilon)}{3[y(\alpha+2)-3\varepsilon]} > 0 \quad \text{for FPIII.} \end{aligned} \quad (6.22)$$

From the inequalities (6.22) it follows that fixed points with the coordinates (5.19) and (5.21) and $w_* = 1$ are IR attractive in the full space of couplings g_1, g_2, u, v, w and governs the IR asymptotic behavior of the full-scale model (6.3).

The critical dimensions of the passive fields θ and θ' are obtained from the Table I and the expression (5.41) for Δ_ω . For fixed point FPIII they are the same as in the case $d = 3$, namely

$$\Delta_\theta = -1 + y/6, \quad \Delta_{\theta'} = d + 1 - y/6. \quad (6.23)$$

For another point FPII they are

$$\Delta_\theta = -1 + \varepsilon/4, \quad \Delta_{\theta'} = d + 1 - \varepsilon/4. \quad (6.24)$$

D. Renormalization and critical dimensions of composite operators

In the following, the central role will be played by composite fields (“composite operators”) built solely of the basic fields θ :

$$F(x) = \theta^n(x). \quad (6.25)$$

In general, a local composite operator is a polynomial constructed from the primary fields $\Phi(x)$ and their finite-order derivatives at a single space-time point $x = (t, \mathbf{x})$. Due to coincidence of the field arguments, new UV divergences arise in the Green functions with such objects [9, 10].

The total canonical dimension of arbitrary 1-irreducible Green function $\Gamma = \langle F \Phi \dots \Phi \rangle$ that includes one composite operator F and arbitrary number of primary fields Φ (the formal index of UV divergence) is given by the relation

$$d_\Gamma = d_F - \sum_{\Phi} N_\Phi d_\Phi, \quad (6.26)$$

where N_Φ are the numbers of the fields entering into Γ , d_Φ are their total canonical dimensions, d_F is the canonical

dimension of the operator, and the summation over all types of the fields is implied.

During renormalization procedure operators can mix with each other,

$$F_i = \sum_j Z_{ij} F_j^R, \quad (6.27)$$

and Z_{ij} is the renormalization matrix. But in the simplest case of operators (6.25) matrix Z_{ij} is diagonal, i.e., $F(x) = Z_F F^R(x)$. In particular this means that the critical dimension of the operator is given by the expression (5.41).

Superficial UV divergences, whose removal requires counterterms, can be present only in those functions Γ for which the index of divergence $d_{\Gamma_{N_\Phi}}$ is a non-negative integer. For the operators of the form (6.25) one has $d_F = -n$. Due to the linearity of our model in θ , the number of the fields θ in any 1-irreducible function with the operator $F(x)$ cannot exceed their number in the operator itself. From the analysis of expression (6.26) it therefore follows that the superficial divergence can only be present in the 1-irreducible function with $N_\theta = n$ and $N_\Phi = 0$ for all other types of the fields Φ . For this function $\delta_\Gamma = 0$ and the corresponding counterterm has the form $\theta^n(x)$; hence, operators (6.25) are multiplicatively renormalizable, $F(x) = Z_n F^R(x)$.

Introduce $\Gamma_n(x; \theta)$: the θ^n term of the expansion in $\theta(x)$ of the generating functional of the 1-irreducible Green functions with one composite operator $F(x)$ and any number of fields θ :

$$\begin{aligned} \Gamma_n(x; \theta) &= \int dx_1 \dots \int dx_n \langle F(x) \theta(x_1) \dots \theta(x_n) \rangle \\ &\times \theta(x_1) \dots \theta(x_n). \end{aligned} \quad (6.28)$$

Renormalization constants Z_n are determined by the requirement that the 1-irreducible functions (6.28) be UV finite in renormalized theory.

The one-loop approximation for the 1-irreducible function $\Gamma_n(x; \theta)$ can be formally written as

$$\Gamma_n(x; \theta) = F(x) + \frac{1}{2} \text{ (diagram) } \quad (6.29)$$

The first term is the tree (loop-less) approximation, double solid lines with a slash denotes the propagators $\langle \theta\theta' \rangle$, single solid line corresponds to the propagator $\langle v v \rangle$, $1/2$ is the symmetry coefficient of the given graph, and the dot with two attached lines in the top of the diagram denotes the operator vertex, i.e., the variational derivative

$$\begin{aligned} V(x; x_1, x_2) &= \delta^2 F(x) / \delta\theta(x_1) \delta\theta(x_2) \\ &= n(n-1) \theta^{n-2}(x) \delta(x-x_1) \delta(x-x_2). \end{aligned} \quad (6.30)$$

The contribution of a specific diagram into the functional (6.29) for any composite operator F is represented in the form

$$\Gamma_n = V \times I \times \theta \dots \theta, \quad (6.31)$$

where V is the vertex factor (6.30), I is the diagram itself, and the product $\theta \dots \theta$ corresponds to external tails.

The divergence of the considering graph is logarithmic, hence one should set all the external frequencies and momenta equal to zero. Therefore the analytical expression of the diagram is given by

$$D_9 = \int \frac{d\omega}{2\pi} \int \frac{d^d \mathbf{k}}{(2\pi)^d} V_a(\mathbf{k}) V_c(-\mathbf{k}) \frac{1}{\omega^2 + w^2 \nu^2 k^4} \times [P_{ac}(\mathbf{k}) \times A(k) + Q_{ac}(\mathbf{k}) \times B(k)], \quad (6.32)$$

where $V_a(\mathbf{k})$ and $V_c(-\mathbf{k})$ are two vertices (6.5); scalar coefficients $A(k)$ and $B(k)$ of the propagator $\langle vv \rangle$ are defined in (4.10) with the replacement of original bare parameters to their renormalized counterparts. As $V_a(\mathbf{k}) P_{ac}(\mathbf{k}) = 0$, only second term in (6.32) gives non-vanishing contribution.

Integration over the frequency gives

$$\int \frac{d\omega}{2\pi} \frac{B(k)}{\omega^2 + w^2 \nu^2 k^4} = \frac{1}{2\nu^3} \frac{1}{uw(u+w)} \frac{1}{k^6}. \quad (6.33)$$

Contracting tensor indices, using (4.17), and collecting all the factors one obtains that expression (6.29) takes form

$$\Gamma_n(x; \theta) = \theta^n(x) \left\{ 1 + \frac{n(n-1)}{4wu(u+w)} \left[\alpha g_1 \left(\frac{\mu}{m} \right)^y \frac{1}{y} + g_2 \left(\frac{\mu}{m} \right)^\varepsilon \frac{1}{\varepsilon} \right] \right\}, \quad (6.34)$$

in notation $g_i \rightarrow g_i C_d$ and up to a finite part and higher-order corrections.

The renormalization constants Z_n are found from the requirement that the renormalized analog $\Gamma_n^R = Z_n^{-1} \Gamma_n$ of the function (6.28) be UV finite in terms of renormalized parameters. In contrast to expressions (4.2) – (4.6), in this case renormalization constants Z_n belong not to some parameters, but to the Green functions itself. This means that using loop expansion (6.29) one finds in fact not renormalization constants Z_n , but inversed one Z_n^{-1} . Taking into account minus sign in the exponent, from (6.34) it follows that in the MS scheme renormalization constants take form

$$Z_n = 1 + \frac{n(n-1)}{4wu(u+w)} \left(\frac{\alpha g_1}{y} + \frac{g_2}{\varepsilon} \right). \quad (6.35)$$

The corresponding anomalous dimensions are

$$\gamma_n = -\frac{n(n-1)}{4wu(u+w)} (\alpha g_1 + g_2), \quad (6.36)$$

with higher-order corrections in g_1 and g_2 .

The critical dimensions of the operators θ^n from the expression (5.41) are readily derived

$$\Delta[\theta^n] = n\Delta_\theta + \gamma_n^*. \quad (6.37)$$

Substituting of fixed-point values FP II and FP III into (6.36) finally gives that for fixed point FP II critical dimensions are

$$\Delta[\theta^n] = -n + \frac{n\varepsilon}{4} - \frac{n(n-1)}{3}\varepsilon; \quad (6.38)$$

for fixed point FP III they differs from the case $d = 3$ and are equal to

$$\Delta[\theta^n] = -n + \frac{ny}{6} - \frac{2n(n-1)}{3} \frac{\alpha y(y-\varepsilon)}{y(2+\alpha) - 3\varepsilon}. \quad (6.39)$$

Both expressions (6.38) and (6.39) suppose higher-order corrections in y and ε . For both cases FP II and FP III the dimensions are negative, i.e., “dangerous” in the sense of operator product expansion [7, 8], and decrease as n grows.

E. Operator product expansion and anomalous scaling

The measurable quantities and, therefore, the objects of interest are equal-time pair correlation functions of two (UV finite) renormalized local composite operators $F_{1,2}(x)$. From the dimensionality considerations (see Table I) it follows that

$$\langle F_1(t, \mathbf{x}_1) F_2(t, \mathbf{x}_2) \rangle = \nu^{d_F^\omega} \mu^{d_F} \times f(\mu r, mr, c/\mu\nu), \quad (6.40)$$

where d_F^ω and d_F are frequency and total canonical dimensions of the correlation function, $r = |\mathbf{x}_2 - \mathbf{x}_1|$, and f is a function of dimensionless variables.

If the correlation function (6.40) is multiplicatively renormalizable, in the IR region it fulfills the differential equation (5.40), which describes the IR scaling behavior. That is, the behavior of the function f for $\mu r \gg 1$ is determined by the IR attractive fixed points FP II and FP III of the RG equation. The solution of this equation leads to the following asymptotic expression:

$$\langle F_1(t, \mathbf{x}_1) F_2(t, \mathbf{x}_2) \rangle \simeq \nu^{d_F^\omega} \mu^{d_F} (\mu r)^{-\Delta_F} \times h[mr, \bar{c}(r)]. \quad (6.41)$$

Here Δ_F is the critical dimension of the correlation function, given by simple sum of the dimensions of the operators; h is unknown scaling function with completely (both canonically and critically) dimensionless arguments, and $\bar{c}(r)$ is invariant speed of sound.

For the composite operator $F(x) = \theta^n(x)$, expression (6.41) yields

$$\langle \theta^p(t, \mathbf{x}_1) \theta^k(t, \mathbf{x}_2) \rangle \simeq \mu^{-(p+k)} (\mu r)^{-\Delta_p - \Delta_k} h_{pk}[mr, \bar{c}(r)], \quad (6.42)$$

where critical dimensions Δ_n for two scaling regimes are given by expressions (6.38) and (6.39).

Representation (6.42) holds for $\mu r \gg 1$ and any fixed value of mr . The inertial-convective range $l \ll r \ll L$ corresponds to the additional condition $mr \ll 1$. The behavior of the functions h at $mr \rightarrow 0$ can be studied by means of the operator product expansion; see [7, 9]. According to the OPE, the equal-time product $F_1(x_1)F_2(x_2)$ of two renormalized operators for $\mathbf{x} \equiv (\mathbf{x}_1 + \mathbf{x}_2)/2 = \text{const}$ and $\mathbf{r} \equiv \mathbf{x}_1 - \mathbf{x}_2 \rightarrow 0$ has the representation

$$F_1(t, \mathbf{x}_1)F_2(t, \mathbf{x}_2) \simeq \sum_F C_F[mr, \bar{c}(r)]F(t, \mathbf{x}), \quad (6.43)$$

where C_F are numerical coefficient functions analytical in mr and $\bar{c}(r)$ and F are all possible renormalized local composite operators allowed by the symmetry.

The correlation function (6.41) is obtained by averaging (6.43) with the weight $\exp \mathcal{S}_R$, where \mathcal{S}_R is the renormalized action functional (6.3). Mean values $\langle F(x) \rangle \propto (mr)^{\Delta_F}$ appear on the right hand side. Their asymptotic behavior at small m is found from the corresponding RG equations and has the form

$$\langle F(x) \rangle \simeq m^{\Delta_F} \times q[\bar{c}(1/m)], \quad (6.44)$$

with another set of scaling functions q . Since the diagrams of the perturbation theory have finite limits both for $c \rightarrow \infty$ and $c \rightarrow 0$, we may assume that the functions $q(c)$ are restricted for all values of c and can be estimated by some constants. Moreover, for invariant variable $\bar{c}(r)$ IR asymptotic behavior together with requirement of its dimensionless gives

$$c(r) = c \times (\mu r)^{\Delta_c} / (\mu \nu), \quad (6.45)$$

where c is renormalized speed of sound. This means that $\bar{c}(1/m) \sim cm^{-\Delta_c}$. Taking into account (5.42), for non-local scaling regime FPIII one obtains that for $y > 12/5$ (i.e., including the most realistic case $y \rightarrow 4$) the argument $cm^{-\Delta_c}$ becomes small for fixed c and $m \rightarrow 0$, and the function q can be replaced by its finite limit value $q(0)$. For local scaling regime FPII from (5.43) it follows that at $\varepsilon \rightarrow 1$ function q can be replaced by its finite limit value $q(\infty)$. From these two remarks we finally conclude that in the IR range for both local and non-local scaling regimes up to a different constants we can write

$$\langle F(x) \rangle \sim m^{\Delta_F}. \quad (6.46)$$

Combining RG representation (6.42) with information gained from OPE (6.43) and expression (6.46) gives the desired asymptotic behavior of the scaling functions:

$$h[mr, c(r)] \simeq \sum_F A_F[mr, c(r)] \times (mr)^{\Delta_F}, \quad (6.47)$$

where the summation runs over all Galilean invariant scalar operators (including operators with derivatives,

etc.), with the coefficient functions A_F analytical in their arguments. The leading contribution into the sum (6.47) is given by the operator with the lowest (minimal) critical dimension; the others can be considered as corrections. The anomalous scaling (i.e., singular behavior as $mr \rightarrow 0$) results from the contributions of the operators with negative critical dimensions. From (6.38) and (6.39) it is easily seen that for both scaling regimes all the operators θ^n have negative dimensions, and the spectrum of their dimensions is not restricted from below.

Fortunately, due to the linearity of initial stochastic equation (6.1) in the field θ , the number of such fields on the right hand side of expression (6.43) cannot exceed their number on the left hand side. This means that for a given correlation function only a finite number of those operators can contribute to the OPE. For correlation functions (6.42) these operators are those for which $n \leq p + k$. The leading term of the behavior as $mr \rightarrow 0$ is given by the operator with the maximum possible $n = p + k$ and without any derivatives, so the final expression has the form

$$\langle \theta^p(t, \mathbf{x}_1) \theta^k(t, \mathbf{x}_2) \rangle \simeq \mu^{-(p+k)} (\mu r)^{-\Delta_p - \Delta_k} (mr)^{\Delta_{p+k}}. \quad (6.48)$$

The fact that the leading term in the OPE is given by the operator from the same family with the summed exponent together with inequality $\Delta_p + \Delta_k > \Delta_{p+k}$ can be interpreted as the statement that the correlations of the scalar field in the model (6.1) show multifractal behavior; see [56].

VII. CONCLUSION

In this paper, which is a natural extension of [45], the stochastic Navier Stokes equation for a *compressible* fluid has been studied using the field theoretic approach. In contrast to previous work, the model is considered in special space dimension $d = 4$. In this case additional UV divergence is observed in 1-irreducible Green function $\langle v'v' \rangle$. This fact significantly affects RG analysis, regarding both technical aspects and results. In particular, renormalization group technique with double expansion scheme is needed. In the one loop approximation, depending on the exponent y and deviation from the dimension of \mathbf{x} space $\varepsilon = 4 - d$, the model possesses two stable nontrivial fixed points in the IR region (i.e., two possible nontrivial scaling regimes) – local one, called in text FPII, and nonlocal, called FPIII.

This shows that the simple analysis around $d = 3$, which indicates existence of only one nontrivial fixed point, corresponding to non-local scaling regime, can be regarded as incomplete in this case. The crossover between local and non-local regimes occurs along the line $y = 3\varepsilon/2$, which is in accordance with [28]. The new (local) regime, which takes place at $d = 4$, should by continuity move into $d = 3$ at $\varepsilon \rightarrow 1$. Nevertheless, at physical values $\varepsilon = 1$ and $y = 4$ new local regime FPII is

unstable, and non-local regime FPIII is realized, therefore the main conclusions of the previous papers [29, 45] about existence and IR attraction of this fixed point remain true.

The model of passive scalar advection of density field by this velocity ensemble has been also analyzed. The full stochastic problem can be formulated as field theoretic model, which appear multiplicatively renormalizable. The new parameter κ does not affect to RG functions of the Navier-Stokes equation itself, so the critical behavior of this model is also described by two fixed points, local one and non-local one.

The inertial range ($l \ll r \ll L$) behavior of correlation functions has been studied by means of the OPE technique. Existence of anomalous scaling, i.e., singular power-like dependence on the integral scale L , has been established. The corresponding anomalous exponents have been identified with critical dimensions of certain composite operators and calculated in the leading one-loop approximation.

As in the case $d = 3$, results of the present model have some similarities and some differences with simplified models, which describe compressible turbulent velocity field as some Gaussian statistical ensemble [37–41]. In contrast to [40, 41] (Gaussian ensemble with finite correlation time), the present model is manifestly Galilean covariant, and this fact holds in all orders of the perturbation theory. As in the rapid-change (zero correlation time) models, critical dimensions are insensitive to the specific choice of the random force. This is a consequence of the fact that propagator $\langle \theta\theta \rangle_0$ does not enter into the relevant Green functions.

In principle, the anomalous exponents could depend on some dimensionless parameters like u_0 , v_0 , or w_0 . After the RG procedure, these parameters are replaced with the corresponding invariant variables. Existence of the unique IR attractive fixed point for each pair y, ε shows that in the IR range these invariant variables tend to their fixed-point values u_* , v_* , w_* irrespective of the initial values u_0 , v_0 , and w_0 . This is a consequence of the one-loop approximation; in higher-loop level overlaps may occur in the picture of domains of IR stability.

In contrast to the case $d = 3$, at $d = 4$ anomalous dimensions of composite operators (6.38) and (6.39) do not grow with α without bound. This is a consequence

of eliminating of poles in ε near $d = 4$, which leads to a significant improvement of situation near $d = 3$. The situation is similar to consideration of Navier-Stokes equation for incompressible fluid near $d = 2$, where renormalization procedure at $d = 2$ is a reason for the Kolmogorov constant's better agreement with experimental results [50].

As in the case $d = 3$, at $d = 4$ the coordinates of both nontrivial fixed points are valid for all $\alpha > 0$, have finite limits at $\alpha \rightarrow \infty$, but ceases to exist for purely potential forcing, i.e., for $\alpha = \infty$. This fact means that under the one-loop approximation both fixed points FPII and FPIII appears to disappear or lose its stability. Corresponding scaling regimes consequently undergoes some qualitative changeover, which is possibly accompanied by phase transition to a purely chaotic state observed in [36] for a simplified model.

To clarify this problem it is necessary to go beyond the one-loop approximation and to discuss the existence, stability and the dependence on α of fixed points at least at the two-loop level, which seems to be a difficult technical task. Also it would be very interesting to investigate a scalar admixture in case of tracer field, or passively advected vector fields. Another very important task is to develop compressible Navier-Stokes equation near $d = 2$. Such analysis may show another types of IR behavior or another dependence of different parameters like α , viscosity ratios, etc. This work is left for the future and is partly in progress.

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